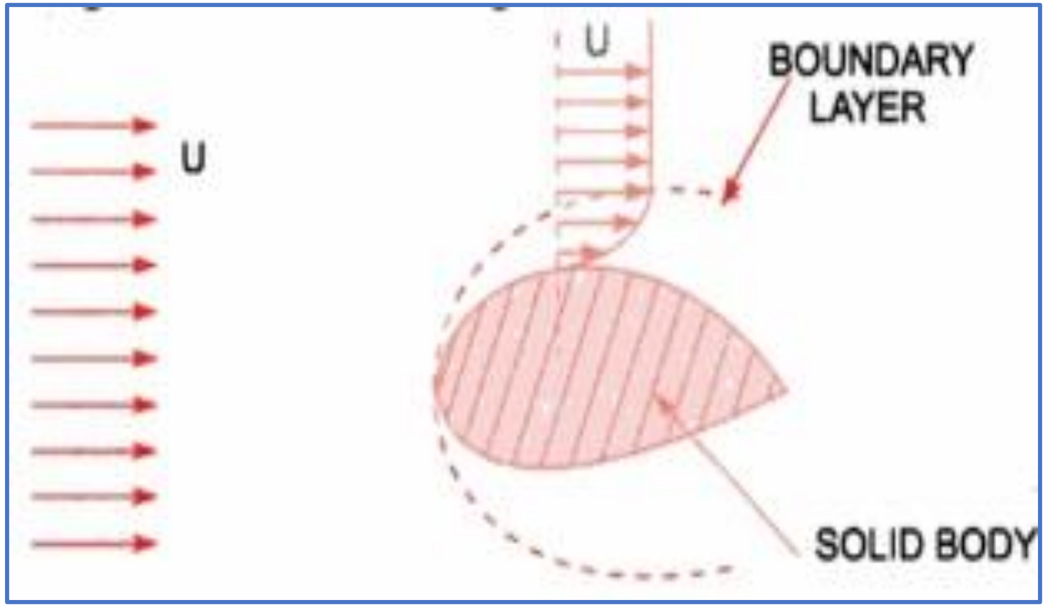
Boundary layer theory

Boundary layer theory

- When a real fluid flows past a solid body or a solid wall, the fluid particles adhere to the boundary and condition of no slip occurs.
- This means that the velocity of fluid close to the boundary will be same as that of the boundary.



- If the boundary is stationary, the velocity of fluid at the boundary will be zero.
- Farther away from the boundary, the velocity will be higher and as a result of this variation of velocity, the velocity gradient & will exist.

- Away from the boundary the velocity increases gradually and reaches free stream velocity at some distance from the boundary and here there is a small region close to the boundary where velocity gradient exist and this region is known as Boundary layer region.
- In the boundary layer region the flow is **highly viscous** and hence Bernoulli's equation is not valid or not applicable in boundary layer region

- Due to boundary of small region is known as **BLR** the velocity decreases due to boundary and there are losses.

- Due to boundary layer formations losses are arises but in the rivers, large diameter pipes ….etc we neglect but in small diameters boundary layer causes losses in properties like viscosity.

Boundary layer theory

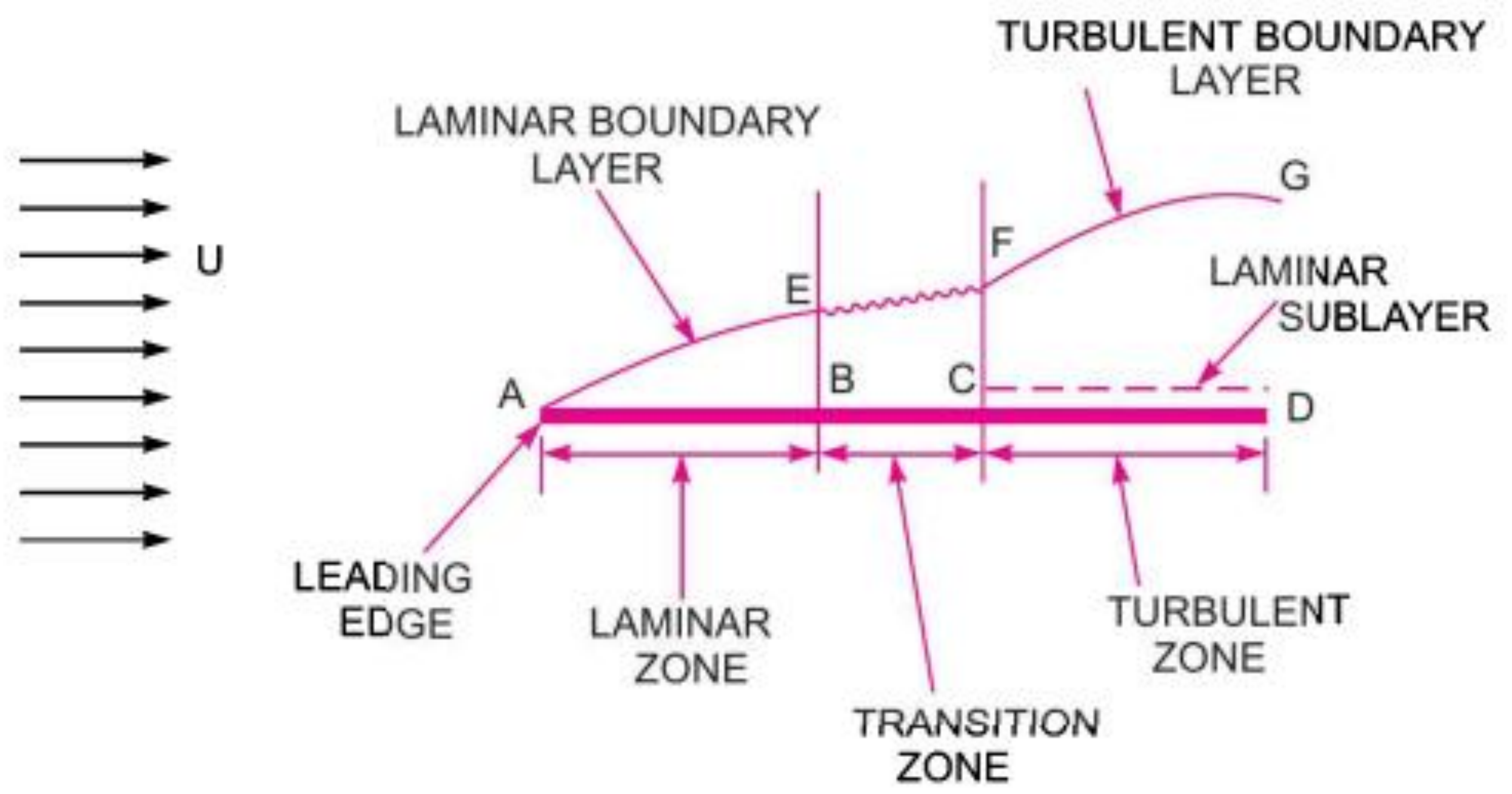
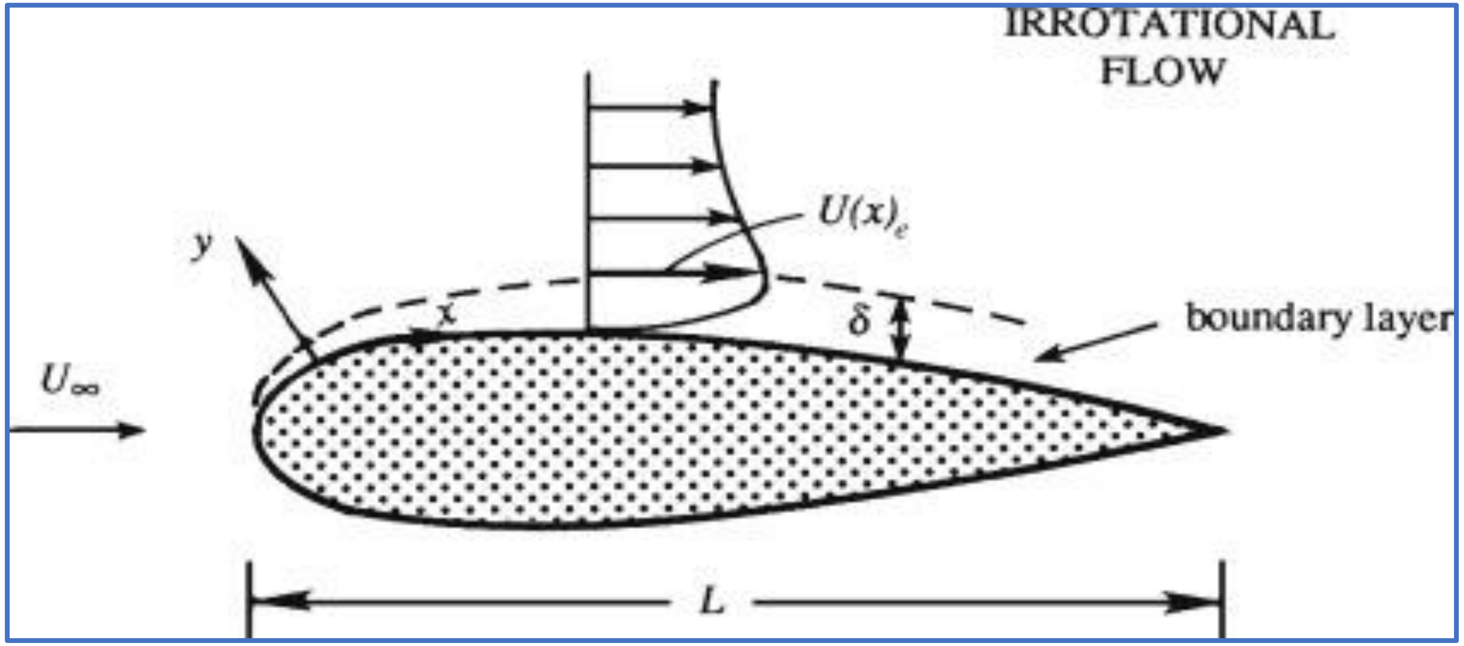


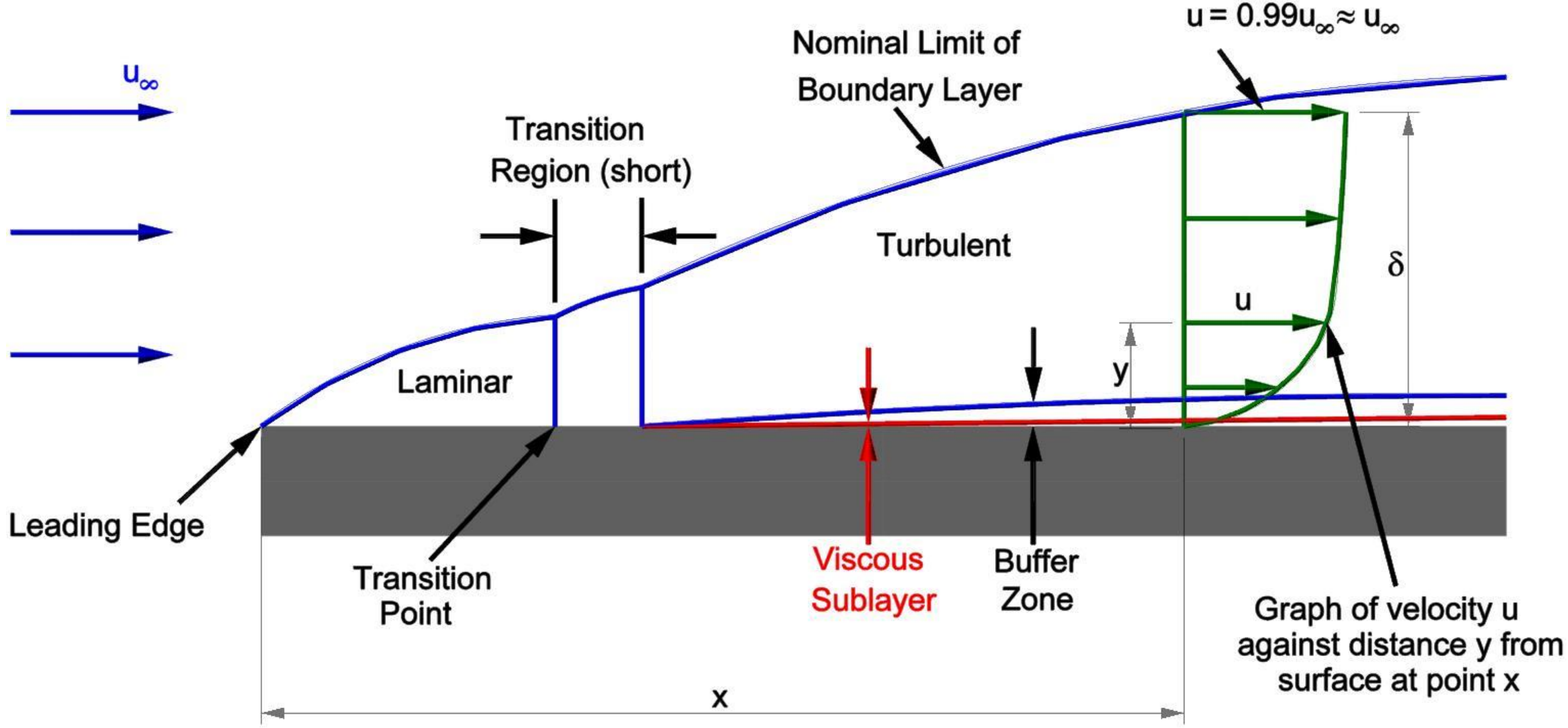
Fig. 13.2 *Flow over a plate.*



IRROTATIONAL FLOW
$U(x)_e$
y
x
δ
boundary layer
U_∞
L

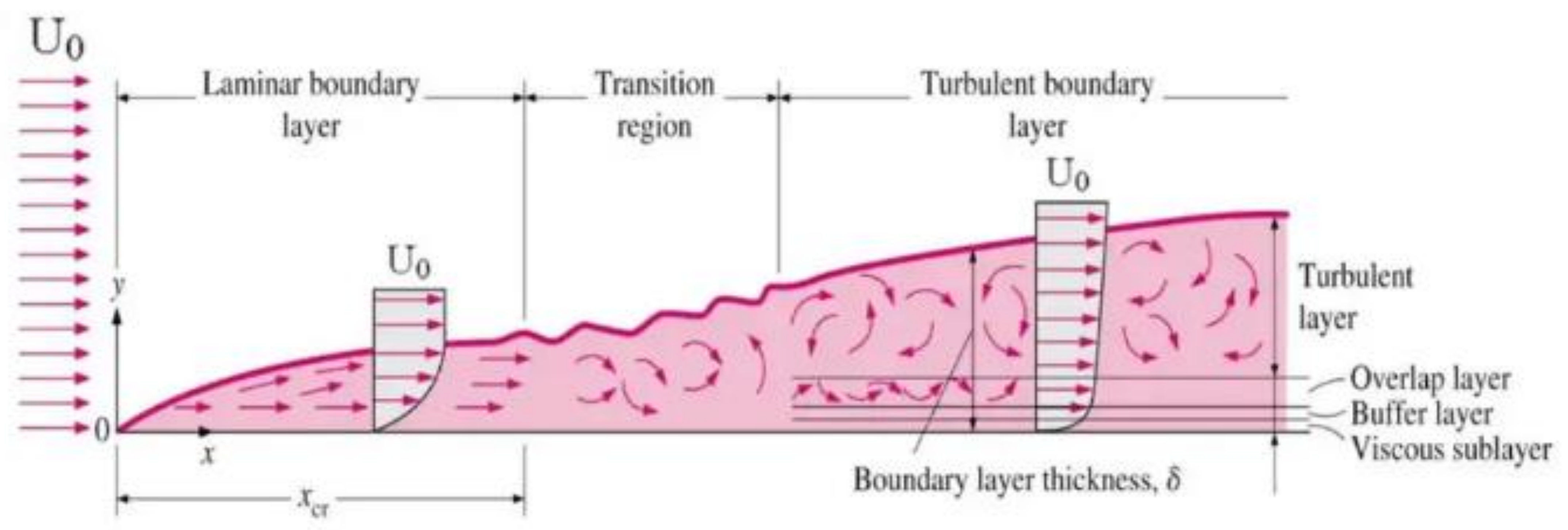
Growth of boundary layer over a flat plate:

- When ever a real fluid flows pass a flat plate the velocity of fluid on the plate will be same as that of plate velocity.
- If the plate at rest ,the fluid will also have zero velocity and BLR will grow distance from the leading edge
- Up to certain distance BLR is laminar as the distance from the leading edge increases the laminar BLR grows in instability and flow changes from laminar to turbulent



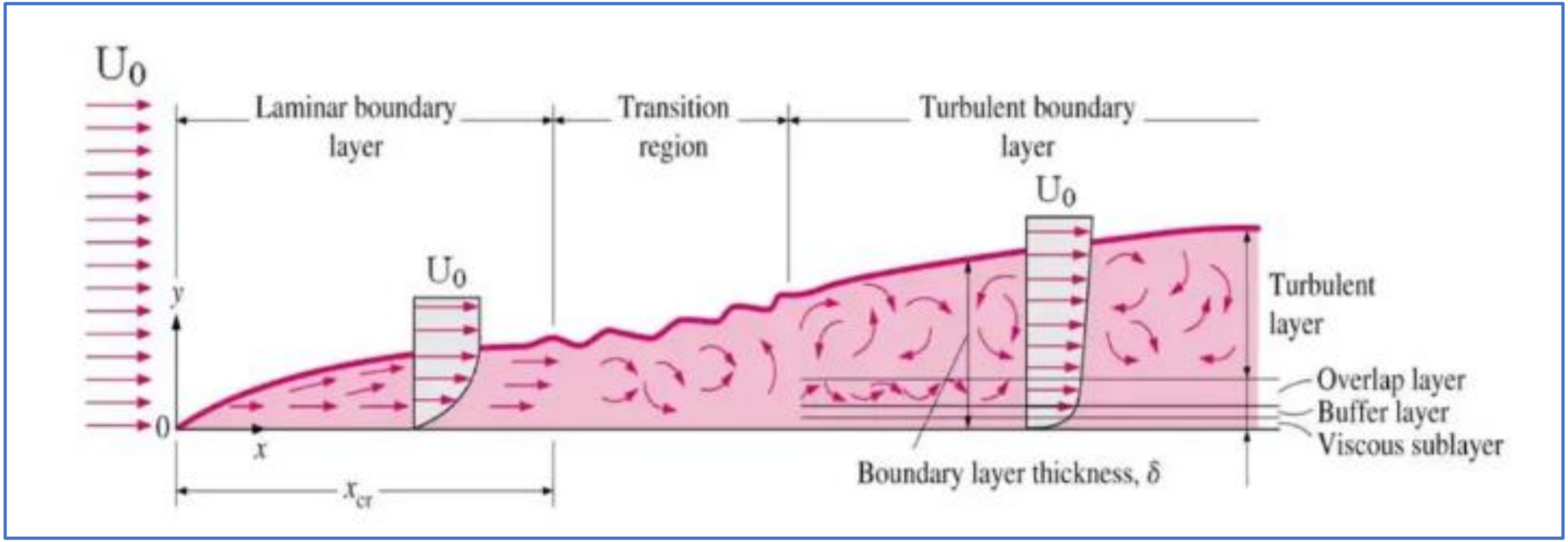
u = 0.99u∞ ≈ u∞
Nominal Limit of
Boundary Layer
u∞
Transition
Region (short)
Turbulent
δ
u
y
Laminar
Leading Edge
Transition
Point
Viscous
Sublayer
Buffer
Zone
Graph of velocity u
against distance y from
surface at point x
x

Growth of boundary layer over a flat plate:



It is found the even turbulent region close to the plate the flow is laminar and the region is known as **laminar sub layer.** Laminar sub layer exist in turbulent boundary layer region.

Boundary layer theory



Boundary conditions :

At y=0 ;u=0

At y= δ ;U =U_∞

At y= δ ; $\frac{du}{dy} = 0$

At x =0 ; δ =0

Boundary layer thickness(δ) :

It is the vertical distance friom the boundary till the point where the velocity becomes 99 % of the free stream velocity

At At y= δ ;U = $0.99U_\infty$

But for the numerical we assume **At y= δ ;U =**U_∞

Displacement thickness (δ^*):

It is defined as the distance by which the solid boundary must be shifted in case of ideal fluid to compensate for the loss in mass flow rate due to boundary layer region

$$\boldsymbol{\delta}^* = \int_0^{\boldsymbol{\delta}} \left(1 - \frac{\boldsymbol{U}}{\boldsymbol{U}_\infty}\right)$$

Momentum thickness :

It is defined as the distance by which the sloid body must be shifted in case of ideal fluid to compensate for the loss in momentum due to boundary layer region

$$\theta = \int_0^{\delta} \frac{U}{U_\infty}\left(1 - \frac{U}{U_\infty}\right) dy$$

Energy thickness δ^{**} :

It is defined by the distance by which the distance by which the solid boundary must be shifted in case of ideal fluid to compensate for the loss in energy due to boundary layer region.

$$\delta^{**} = \int_0^{\delta} \frac{U}{U_\infty}\left(1 - \left(\frac{U}{U_\infty}\right)^2\right) dy$$

Problem 13.1 *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $\frac{u}{U} = \frac{y}{\delta}$*, where u is the velocity at a distance y from the plate and u = U at y = δ, where δ = boundary layer thickness. Also calculate the value of δ*/θ.*

Solution. Given :

Velocity distribution $\frac{u}{U} = \frac{y}{\delta}$

(*i*) Displacement thickness δ* is given by equation (13.2),

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy \qquad \left\{\because \frac{u}{U} = \frac{y}{\delta}\right\}$$

$$= \left[y - \frac{y^2}{2\delta}\right]_0^\delta \qquad \{\delta \text{ is constant across a section}\}$$

$$= \delta - \frac{\delta^2}{2\delta} = \delta - \frac{\delta}{2} = \frac{\delta}{2}. \quad \textbf{Ans.}$$

(*ii*) Momentum thickness, θ is given by equation (13.5),

$$\theta = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = \frac{y}{\delta}$,

$$\theta = \int_0^{\delta} \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \int_0^{\delta} \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy$$

$$= \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^{\delta} = \frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2} = \frac{\delta}{2} - \frac{\delta}{3} = \frac{3\delta - 2\delta}{6} = \boldsymbol{\frac{\delta}{6}}.\ \textbf{Ans.}$$

(*iii*) Energy thickness δ^{**} is given by equation (13.6), as

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1 - \frac{u^2}{U^2}\right] dy = \int_0^\delta \frac{y}{\delta}\left[1 - \frac{y^2}{\delta^2}\right] dy \qquad \left\{\because \quad \frac{u}{U} = \frac{y}{\delta}\right\}$$

$$= \int_0^\delta \left[\frac{y}{\delta} - \frac{y^3}{\delta^3}\right] dy = \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3}$$

$$= \frac{\delta}{2} - \frac{\delta}{4} = \frac{2\delta - \delta}{4} = \boldsymbol{\frac{\delta}{4}}. \textbf{ Ans.}$$

(*iv*)

$$\frac{\delta^*}{\theta} = \frac{\left(\frac{\delta}{2}\right)}{\left(\frac{\delta}{6}\right)} = \frac{\delta}{2} \times \frac{6}{\delta} = \textbf{3. Ans.}$$

Problem 13.2 *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$.

Solution. Given :

Velocity distribution $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

(*i*) Displacement thickness δ^* is given by equation (13.2),

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$, we have

$$\delta^* = \int_0^\delta \left\{1 - \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right\} dy$$

$$= \int_0^\delta \left\{1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right\} dy = \left[y - \frac{2y^2}{2\delta} + \frac{y^3}{3\delta^2}\right]_0^\delta$$

$$= \delta - \frac{\delta^2}{\delta} + \frac{\delta^3}{3\delta^2} = \delta - \delta + \frac{\delta}{3} = \frac{\delta}{3}. \quad \textbf{Ans.}$$

(*ii*) Momentum thickness θ, is given by equation (13.5),

$$\theta = \int_0^\delta \frac{u}{U}\left\{1 - \frac{u}{U}\right\} dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left[1 - \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right]\left[1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4}\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4}\right] dy = \left[\frac{2y^2}{2\delta} - \frac{5y^3}{3\delta^2} + \frac{4y^4}{4\delta^3} - \frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \left[\frac{\delta^2}{\delta} - \frac{5\delta^3}{3\delta^2} + \frac{\delta^4}{\delta^3} - \frac{\delta^5}{5\delta^4}\right] = \delta - \frac{5\delta}{3} + \delta - \frac{\delta}{5}$$

$$= \frac{15\delta - 25\delta + 15\delta - 3\delta}{15} = \frac{30\delta - 28\delta}{15} = \mathbf{\frac{2\delta}{15}}.$$ **Ans.**

(*iii*) Energy thickness δ^{**} is given by equation (13.6),

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1 - \frac{u^2}{U^2}\right]dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right]^2\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \left[\frac{4y^2}{\delta^2} + \frac{y^4}{\delta^4} - \frac{4y^3}{\delta^3}\right]\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{4y^2}{\delta^2} - \frac{y^4}{\delta^4} + \frac{4y^3}{\delta^3}\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{8y^3}{\delta^3} - \frac{2y^5}{\delta^5} + \frac{8y^4}{\delta^4} - \frac{y^2}{\delta^2} + \frac{4y^4}{\delta^4} + \frac{y^6}{\delta^6} - \frac{4y^5}{\delta^5}\right) dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{y^2}{\delta^2} - \frac{8y^3}{\delta^3} + \frac{12y^4}{\delta^4} - \frac{6y^5}{\delta^5} + \frac{y^6}{\delta^6}\right]dy$$

$$= \left[\frac{2y^2}{2\delta} - \frac{y^3}{3\delta^2} - \frac{8y^4}{4\delta^3} + \frac{12y^5}{5\delta^4} - \frac{6y^6}{6\delta^5} + \frac{y^7}{7\delta^6}\right]_0^\delta$$

$$= \frac{\delta^2}{\delta} - \frac{\delta^3}{3\delta^2} - \frac{2\delta^4}{\delta^3} + \frac{12\delta^5}{5\delta^4} - \frac{\delta^6}{\delta^5} + \frac{\delta^7}{7\delta^6} = \delta - \frac{\delta}{3} - 2\delta + \frac{12}{5}\delta - \delta + \frac{\delta}{7}$$

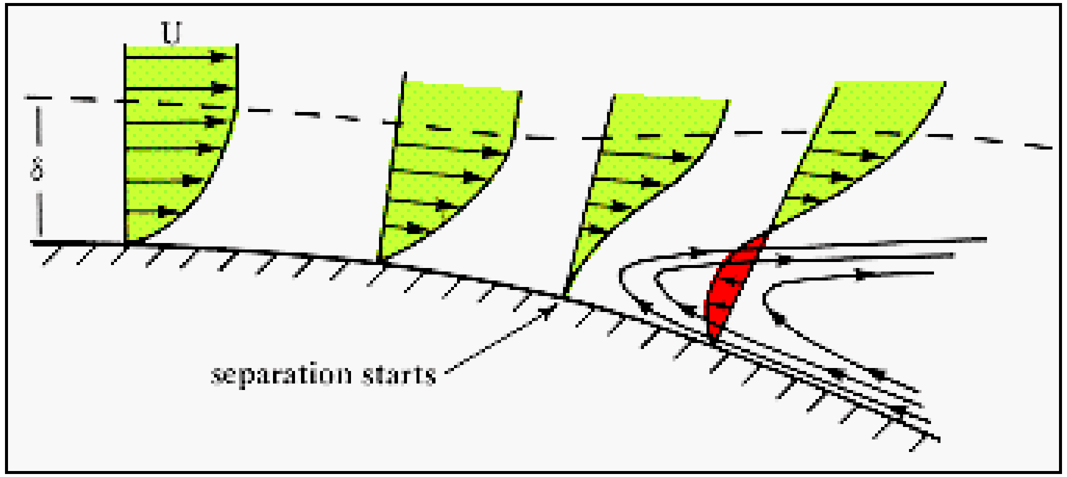
$$= -2\delta - \frac{\delta}{3} + \frac{12}{5}\delta + \frac{\delta}{7} = \frac{-210\delta - 35\delta + 252\delta + 15\delta}{105}$$

$$= \frac{-245\delta + 267\delta}{105} = \mathbf{\frac{22\delta}{105}}.$$ **Ans.**

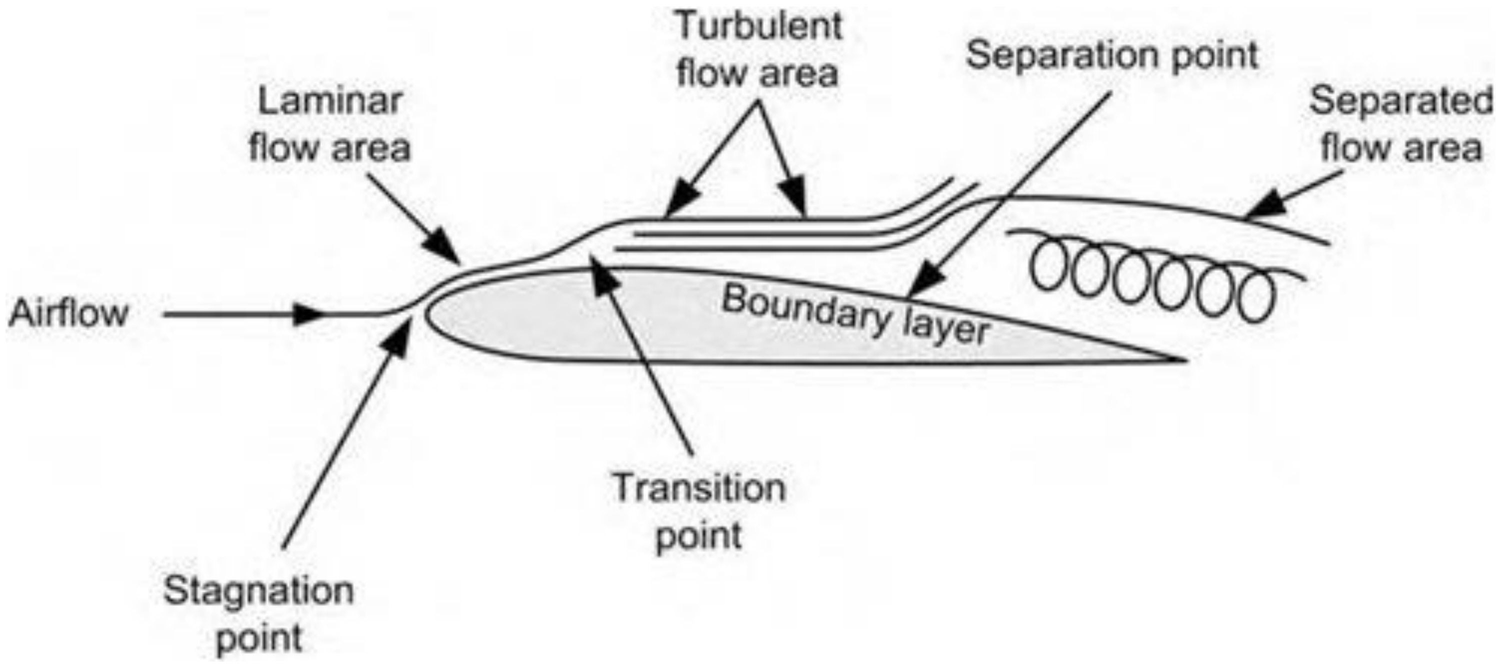
Boundary layer separation:

- When the real fluid flows in converging passage the velocity increases and the pressure decrease and hence fluid flows under negative pressure gradient this flow is also known as accelerating flow.

- Hence boundary layer thickness decreases there fore negative pressure gradients are known as favourable pressure gradient



- When the fluid flows through diverging pipe the velocity decreases and pressure increases the fluid under positive gradient



Boundary layer separation due to positive pressure gradient

- If the angle at the divergence is large the retardation of fluid particles is more and at the same point the moment of the fluid particles may not support the flow and **flow might be separate from its boundary** and reverse in flow deviation is known as boundary layer separation.

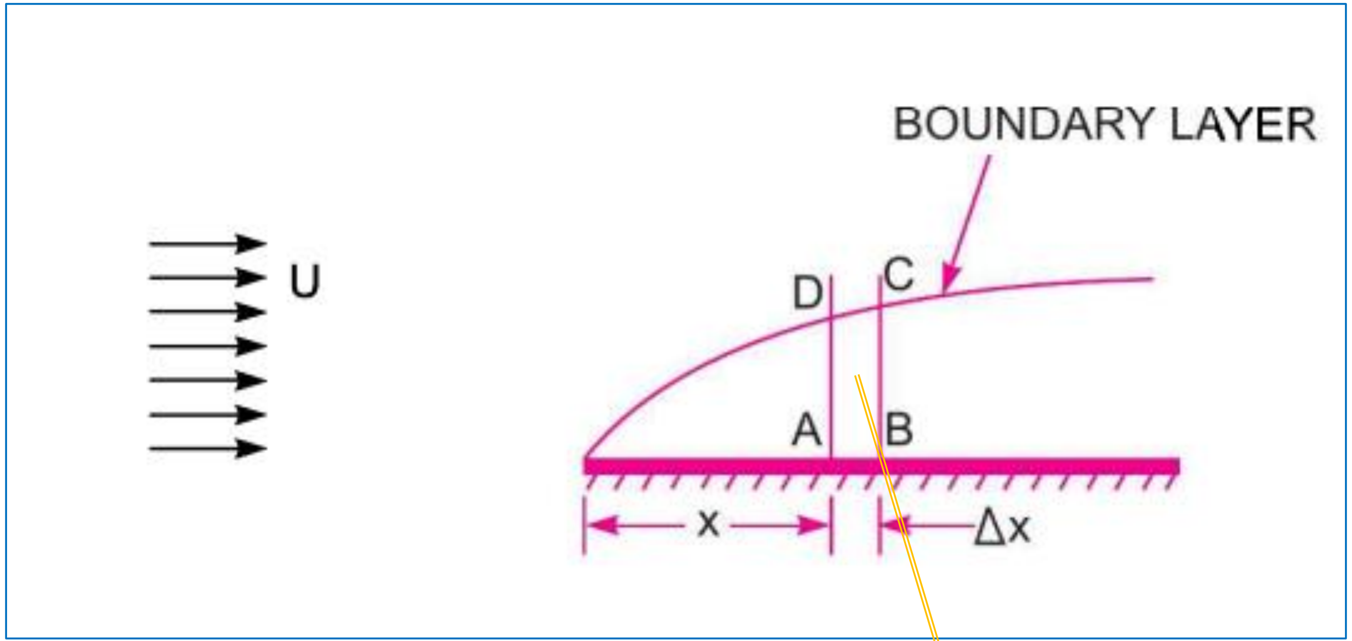
Methods to control boundary layer separation:

1. accelerating the fluid in the diverging region .
2. Streamlining the body
3. Sucking the fluid in the boundary layer region
4. Keeping the divergence angle to a minimum

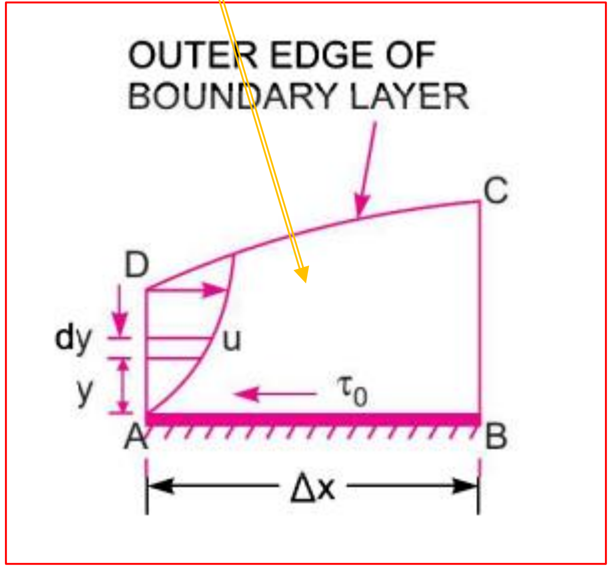

AirShaper

Moment equation for boundary layer by Von Karman (Explanation only)

- Consider the flow of a fluid having free stream velocity equal to **U** ,over a thin plate.
- the drag force on the plate can be determined if the velocity profile near the plate is known



Consider a small length Δx opf the plate at a distance of x from the lading edge the enlarged view of the small length of the plate

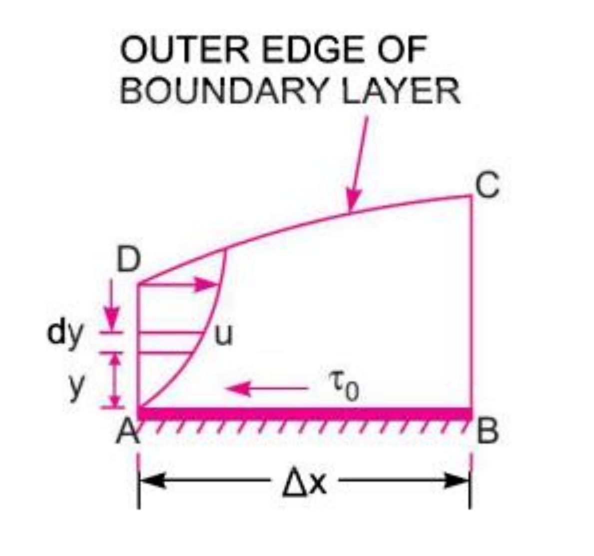


The shear stress near the plate $\boldsymbol{\tau_0} = \mathbf{\mu}\left(\frac{du}{dy}\right)_{y=0}$

Where $\left(\frac{du}{dy}\right)_{y=0}$ is the velocity distribution near the plate at y = 0

ΔF_D= drag force on distance Δx

ΔF_D= drag force = the rate of change of momentum over the distance Δx



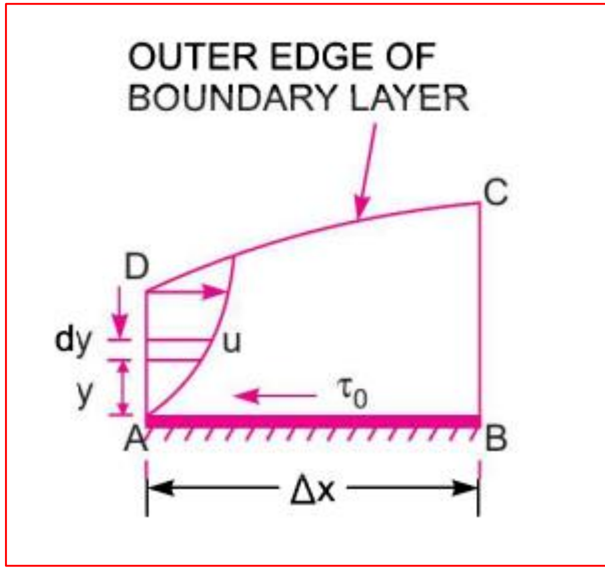
Let ABCD is the control volume of the fluid over the distance Δx

Edge DC = Outer edge of the boundary layer

u= velocity at any within the boundary layer

b= width of the plate

Then mass rate of flow entering through the side AD



$$= \int_0^\delta \rho \times \text{velocity} \times \text{area of strip of thickness } dy$$

$$= \int_0^\delta \rho \times u \times b \times dy \qquad \{\because \quad \text{Area of strip} = b \times$$

$$= \int_0^\delta \rho u b dy$$

Mass rate of flow leaving the side BC

$$= \text{mass through } AD + \frac{\partial}{\partial x}(\text{mass through } AD) \times \Delta x$$

$$= \int_0^\delta \rho u b dy \ \frac{\partial}{\partial x}\left[\int_0^\delta (\rho u b dy)\right] \times \Delta x$$

From continuity equation for a steady incompressible fluid flow, we have

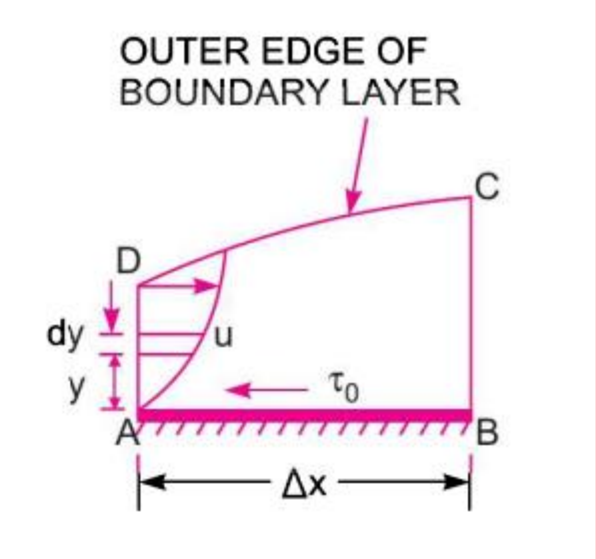
Mass rate of flow entering AD + mass rate of flow entering DC

= mass rate of flow leaving BC

$\because$ Mass rate of flow entering DC = mass rate of flow through BC – mass rate of flow through AD

$$= \int_0^\delta \rho u b dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u b dy\right] \times \Delta x - \int_0^\delta \rho u b dy$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u b dy\right] \times \Delta x$$



The fluid is entering through side DC with a uniform velocity U.

Now let us calculate momentum flux through control volume.

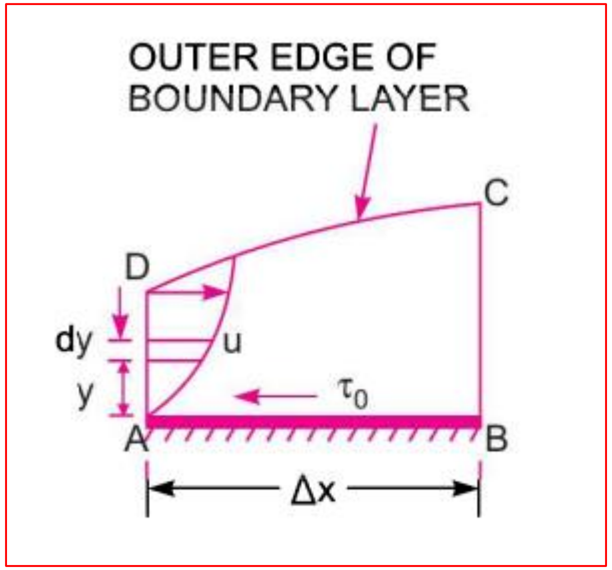
Momentum flux entering through AD

$$= \int_0^\delta \text{momentum flux through strip of thickness } dy$$

$$= \int_0^\delta \text{mass through strip} \times \text{velocity} = \int_0^\delta (\rho u b dy) \times u = \int_0^\delta \rho u^2 b dy$$

Momentum flux leaving the side BC = $\int_0^\delta \rho u^2 b dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy\right] \times \Delta x$

Momentum flux leaving the side $BC = \int_0^\delta \rho u^2 b dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy\right] \times \Delta x$



Momentum flux entering the side DC = mass rate through $DC \times$ velocity

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u b dy\right] \times \Delta x \times U$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u U b dy\right] \times \Delta x$$

As U is constant and so it can be taken inside the differential and integral.

∴ Rate of change of momentum of the control volume

= Momentum flux through BC – Momentum flux through AD

– momentum flux through DC

$$= \int_0^\delta \rho u^2 b dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy\right] \times \Delta x - \int_0^\delta \rho u^2 b dy - \frac{\partial}{\partial x}\left[\int_0^\delta \rho u U b dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy\right] \times \Delta x - \frac{\partial}{\partial x}\left[\int_0^\delta \rho u U b dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy - \int_0^\delta \rho u U b dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\rho u^2 b - \rho u U b\right) dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\rho b \int_0^\delta \left(u^2 - uU\right) dy\right] \times \Delta x$$

{For incompressible fluid ρ is constant}

$$= \rho b \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right) dy\right] \times \Delta x$$

Now the rate of change of momentum on the control volume *ABCD* must be equal to the total force on the control volume in the same direction according to the momentum principle. But for a flat plate $\frac{\partial p}{\partial x} = 0$, which means there is no external pressure force on the control volume. Also the force on the side *DC* is negligible as the velocity is constant and velocity gradient is zero approximately. The only external force acting on the control volume is the shear force acting on the side *AB* in the direction from *B* to *A* as shown in Fig. 13.4 (*b*). The value of this force is given by equation (13.7) as

$$\Delta F_D = \tau_0 \times \Delta x \times b$$

∴ Total external force in the direction of rate of change of momentum

$$= -\tau_0 \times \Delta x \times b$$

$$-\tau_0 \times \Delta x \times b = \rho b \frac{\partial}{\partial x}\left[\int_0^{\delta} \left(u^2 - uU\right) dy\right] \times \Delta x$$

Cancelling $\Delta x \times b$, to both sides, we have

$$-\tau_0 = \rho \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right)dy\right]$$

or

$$\tau_0 = -\rho \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right)dy\right] = \rho \frac{\partial}{\partial x}\left[\int_0^\delta \left(uU - u^2\right)dy\right]$$

$$= \rho \frac{\partial}{\partial x}\left[\int_0^\delta U^2\left(\frac{u}{U} - \frac{u^2}{U^2}\right)dy\right] = \rho U^2 \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right]$$

or

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right]$$

$$\boxed{\frac{\tau_0}{\rho U^2} = \frac{\partial \theta}{\partial x}}$$ **Von Karman momentum integral equation**

Von Karman momentum integral equation applicable for :

This is applied to :

1. Laminar boundary layers,
2. Transition boundary layers, and
3. Turbulent boundary layer flows.

$$\frac{\tau_0}{\rho U^2} = \frac{\partial \theta}{\partial x}$$

$$\Delta F_D = \tau_0 \times \Delta x \times b$$

Then total drag on the plate of length L on one side is

$$F_D = \int \Delta F_D = \int_0^L \tau_0 \times b \times dx \quad \{\text{change } \Delta x = dx\}.$$

Local Co-efficient of Drag [C_D*]. It is defined as the ratio of the shear stress τ_0 to the quantity $\frac{1}{2} \rho U^2$. It is denoted by C_D*

Hence $$C_D^* = \frac{\tau_0}{\frac{1}{2}\rho U^2}.$$

where A = Area of the surface (or plate)

U = Free-stream velocity

ρ = Mass density of fluid.

Average Co-efficient of Drag [C_D]. It is defined as the ratio of the total drag force to the quantity $\frac{1}{2} \rho A U^2$. It is also called co-efficient of drag and is denoted by C_D.

Hence $$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2}$$

Boundary Conditions for the Velocity Profiles. The followings are the boundary conditions which must be satisfied by any velocity profile, whether it is in laminar boundary layer zone, or in turbulent boundary layer zone :

1. At $y = 0$, $u = 0$ and $\frac{du}{dy}$ has some finite value
2. At $y = \delta$, $u = U$
3. At $y = \delta$, $\frac{du}{dy} = 0$.

Problem 13.3 *For the velocity profile for laminar boundary layer flows given as*

$$\frac{u}{U} = 2(y/\delta) - (y/\delta)^2$$

find an expression for boundary layer thickness (δ), shear stress (τ_0) and co-efficient of drag (C_D) in terms of Reynold number.

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] = \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]\left[1-\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\right]dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]\left[1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right]dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta}-\frac{4y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy\right]$$

$$= \frac{\partial}{\partial x}\int_0^\delta \left[\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy = \frac{\partial}{\partial x}\left[\frac{2y^2}{2\delta}-\frac{5\times y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{\delta^2}{\delta}-\frac{5}{3}\frac{\delta^3}{\delta^2}+\frac{\delta^4}{\delta^3}-\frac{\delta^5}{5\delta^4}\right] = \frac{\partial}{\partial x}\left[\delta-\frac{5}{3}\delta+\delta-\frac{\delta}{5}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{15\delta-25\delta+15\delta-3\delta}{15}\right] = \frac{\partial}{\partial x}\left[\frac{30\delta-28\delta}{15}\right] = \frac{\partial}{\delta x}\left[\frac{2\delta}{15}\right] = \frac{2}{15}\frac{\partial}{\partial x}\ [\delta]$$

$\therefore$

$$\tau_0 = \rho U^2 \times \frac{2}{15}\frac{\partial}{\partial x}[\delta] = \frac{2}{15}\rho U^2 \frac{\partial[\delta]}{\partial x} \qquad \text{...(13.15)}$$

The shear stress at the boundary in laminar flow is also given by Newton's law of viscosity as

$$\tau_0 = \mu \left(\frac{du}{dy}\right)_{y=0} \qquad \text{...}(ii)$$

But from equation (i), $$u = U\left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right]$$

$$\therefore \qquad \frac{du}{dy} = U\left[\frac{2}{\delta} - \frac{2y}{\delta^2}\right] \qquad \{\because \quad U \text{ is constant}\}$$

$$\therefore \qquad \left(\frac{du}{dy}\right)_{y=0} = U\left[\frac{2}{\delta} - \frac{2\times(0)}{\delta^2}\right] = \frac{2U}{\delta}$$

Equating the two values of τ_0 given by equation (13.15) and (*iii*)

$$\frac{2}{15}\rho U^2 \frac{\partial}{\partial x}[\delta] = \frac{2\mu U}{\delta}$$

or
$$\frac{\delta\partial}{\partial x}[\delta] = \frac{15\mu U}{\rho U^2} = \frac{15\mu}{\rho U} \quad \text{or} \quad \delta\partial[\delta] = \frac{15\mu}{\rho U}\partial x$$

As the boundary layer thickness (δ) is a function of x only.

Hence partial derivative can be changed to total derivative

$$\therefore \qquad \delta d[\delta] = \frac{15\mu}{\rho U}\, dx$$

On integration, we get $$\frac{\delta^2}{2} = \frac{15\mu}{\rho U}\, x + C \qquad \left\{\frac{\mu}{\rho U} \text{ is constant}\right\}$$

$x = 0$, $\delta = 0$ and hence $C = 0$

$$\therefore \qquad \frac{\delta^2}{2} = \frac{15\mu x}{\rho U}$$

$$\therefore \qquad \delta = \sqrt{\frac{2 \times 15\mu x}{\rho U}} = \sqrt{\frac{30\mu x}{\rho U}} = 5.48\sqrt{\frac{\mu x}{\rho U}} \qquad \text{...(13.16)}$$

$$= 5.48\sqrt{\frac{\mu x \times x}{\rho U \times x}} = 5.48\sqrt{\frac{x^2}{R_{e_x}}} \qquad \left\{\because \quad R_{e_x} = \frac{\rho U x}{\mu}\right\}$$

$$= 5.48\,\frac{x}{\sqrt{R_{e_x}}} \qquad \text{...(13.17)}$$

(*ii*) **Shear stress (τ_0) in terms of Reynolds number**

From equation (*iii*), we have $\tau_0 = \dfrac{2\mu U}{\delta}$

Substituting the value of δ from equation (13.17), in the above equation, we get

$$\tau_0 = \frac{2\mu U}{5.48 \dfrac{x}{\sqrt{R_{e_x}}}} = \frac{2\mu U \sqrt{R_{e_x}}}{5.48x} = 0.365 \frac{\mu U}{x} \sqrt{R_{e_x}}$$

(*iii*) **Co-efficient of Drag (C_D)**

From equation (13.14), we have $C_D = \dfrac{F_D}{\frac{1}{2}\rho A U^2}$

where F_D is given by equation (13.12) as

$$F_D = \int_0^L \tau_0 \times b \times dx = \int_0^L 0.365 \frac{\mu U}{x} \sqrt{R_{e_x}} \times b \times dx$$

$$= 0.365 \int_0^L \frac{\mu U}{x} \sqrt{\frac{\rho U x}{\mu}} \times b \times dx \qquad \left\{ \because \ R_{e_x} = \frac{\rho U x}{\mu} \right\}$$

$$= 0.365 \int_0^L \mu U \sqrt{\frac{\rho U}{\mu}} \times \frac{1}{\sqrt{x}} \times b \times dx$$

$$= 0.365\, \mu U \sqrt{\frac{\rho U}{\mu}} \times b \int_0^L x^{-1/2}\, dx = 0.365\, \mu U \sqrt{\frac{\rho U}{\mu}} \times b \times \left[\frac{x^{1/2}}{\frac{1}{2}} \right]_0^L$$

$$= 0.365 \times 2\mu U \sqrt{\frac{\rho U}{\mu}} \times b \times \sqrt{L}$$

$$= 0.73\, b\mu U \sqrt{\frac{\rho U L}{\mu}} \qquad \text{...(13.18)}$$

$$\therefore \qquad C_D = \frac{0.73\, b\mu U \sqrt{\frac{\rho U L}{\mu}}}{\frac{1}{2} \rho A U^2}$$

where A = Area of plate = Length of plate × width = $L \times b$

$$\therefore \qquad C_D = \frac{0.73\, b\mu U}{\frac{1}{2} \rho \times L \times b \times U^2} \sqrt{\frac{\rho U L}{\mu}} = \frac{1.46\, \mu}{\rho L U} \sqrt{\frac{\rho U L}{\mu}}$$

$$= \frac{1.46 \sqrt{\mu}}{\sqrt{\rho U L}} = 1.46 \sqrt{\frac{\mu}{\rho U L}} = \frac{1.46}{\sqrt{R_{e_L}}} \qquad \text{...(13.19)} \qquad \left\{ \because \sqrt{\frac{\mu}{\rho U L}} = \frac{1}{\sqrt{R_{e_L}}} \right\}$$

Table 13.1

	Velocity Distribution	δ	C_D
1.	$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$	$5.48\, x/\sqrt{R_{e_x}}$	$1.46/\sqrt{R_{e_L}}$
2.	$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$	$4.64\, x/\sqrt{R_{e_x}}$	$1.292/\sqrt{R_{e_L}}$
3.	$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$	$5.84\, x/\sqrt{R_{e_x}}$	$1.36/\sqrt{R_{e_L}}$
4.	$\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$	$4.79\, x/\sqrt{R_{e_x}}$	$1.31/\sqrt{R_{e_L}}$
5.	Blasius's Solution	$4.91\, x/\sqrt{R_{e_x}}$	$1.328/\sqrt{R_{e_L}}$

▶ 13.4 TURBULENT BOUNDARY LAYER ON A FLAT PLATE

The thickness of the boundary layer, drag force on one side of the plate and co-efficient of drag due to turbulent boundary layer on a smooth plate at zero pressure gradient are determined as in case of laminar boundary layer provided the velocity profile is known. Blasius on the basis of experiments give the following velocity profile for turbulent boundary layer

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^n \qquad ...(13.35)$$

where $n = \frac{1}{7}$ for $R_e < 10^7$ but more than 5×10^5

$$\therefore \qquad \frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \qquad ...(13.36)$$

Equation (13.36) is not applicable very near the boundary, where the thin laminar sub-layer of thickness δ' exists. Here velocity distribution is influenced only by viscous effects.

The value of τ_0 for flat plate is taken as $\tau_0 = 0.0225\, \rho\, U^2 \left(\frac{\mu}{\rho \delta U}\right)^{1/4}$ ...(13.37)

▶ 13.5 ANALYSIS OF TURBULENT BOUNDARY LAYER

(*a*) If Reynold number is more than 5×10^5 and less than 10^7 the thickness of boundary layer and drag co-efficient are given as :

$$\delta = \frac{0.37x}{\left(R_{e_x}\right)^{1/5}} \text{ and } C_D = \frac{0.072}{\left(R_{e_L}\right)^{1/5}} \quad \text{...(13.44)}$$

where x = Distance from the leading edge

R_{e_x} = Reynold number for length x

R_{e_L} = Reynold number at the end of the plate.

(*b*) If Reynold number is more than 10^7 but less than 10^9, Schlichting gave the empirical equation as

$$C_D = \frac{0.455}{\left(\log_{10} R_{e_L}\right)^{2.58}} \quad \text{...(13.44A)}$$