Fluid Mechanics
and
Hydraulic Machines

Definition:

- Mechanics is the oldest physical science that deals with both stationary and moving bodies under the influence of forces.
- The branch of mechanics that deals with bodies at rest is called **statics**, while the branch that deals with bodies in motion 1s called dynamics.
- The subcategory fluid mechanics 1s defined as the science that deals with the behavior of fluids at rest (fluid statics) or in motion (fluid dynamics), and the interaction of fluids with solids or other fluids at the boundaries.
- The study of fluids at rest is called fluid statics.

Definition:

- The study of fluids in motion, where pressure forces are not considered, is called fluid kinematics and if the pressure forces are also considered for the fluids in motion. that branch of science is called fluid dynamics.
- Fluid mechanics itself is also divided into several categories.
- The study of the motion of fluids that are practically incompressible (such as liquids, especially water, and gases at low speeds) 1s usually referred to as hydrodynamics.
- A subcategory of hydrodynamics is hydraulics, which deals with liquid flows in pipes and open channels.

What is a Fluid?

- A substance exists in three primary phases: solid, liquid, and gas. A substance in the liquid or gas phase is referred to as a fluid.

- Distinction between a solid and a fluid is made on the basis of the substance’s ability to resist an applied shear (or tangential) stress that tends to change its shape.

- A solid can resist an applied shear stress by deforming, whereas a fluid deforms continuously under the influence of shear stress, no matter how small.

- In solids stress is proportional to strain, but in fluids stress is proportional to strain rate. Or rate of shear deformation

What is a Fluid?

- When a constant shear force is applied, a solid eventually stops ,at some fixed strain angle, whereas a fluid strain, never stops deforming and approaches a certain rate of strain.

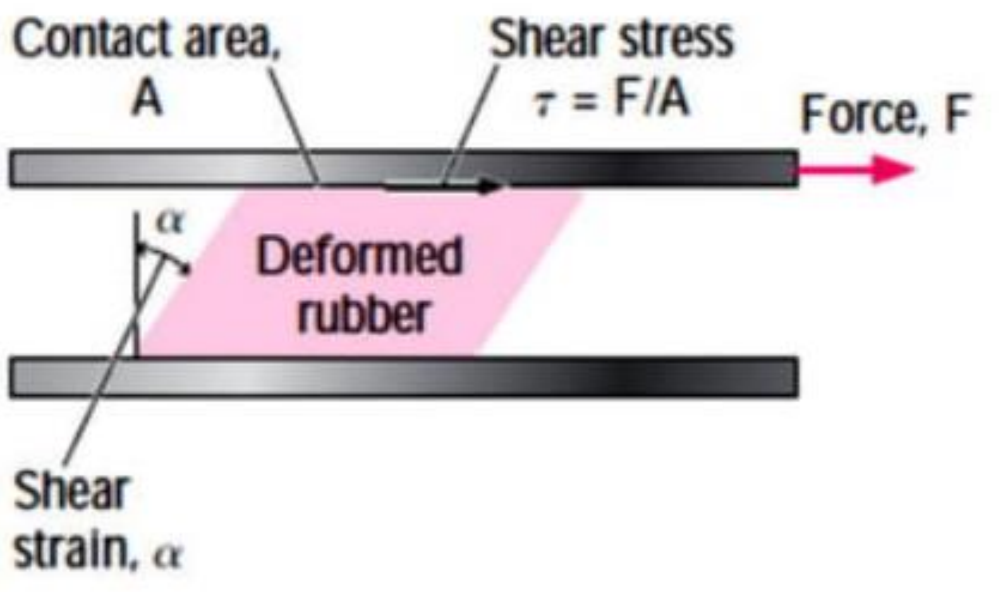
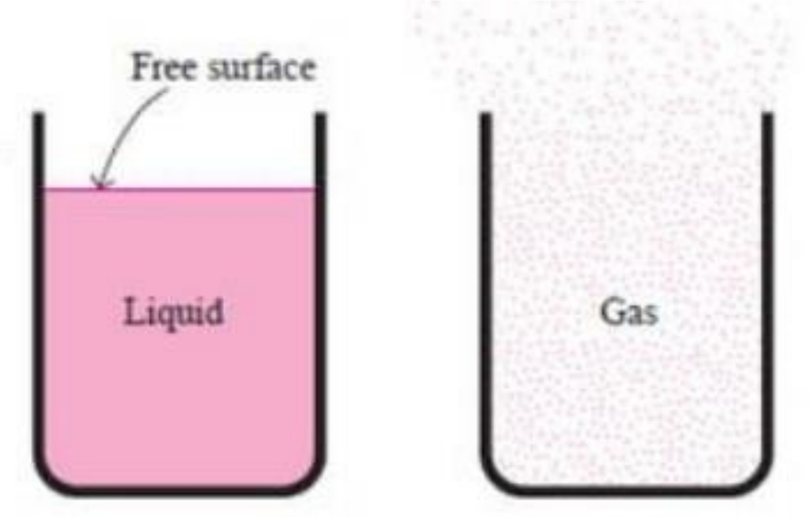


Figure.
Deformation of a rubber eraser placed between two parallel plates under the influence of a shear force.

What is a Fluid?

- In a liquid, molecules can move relative to each other, but the volume remains relatively constant because of the strong cohesive forces between the molecules.



- As a result, a liquid takes the shape of the container it is in, and it forms a free surface in a larger container in a gravitational field.
- A gas, on the other hand, expands until it encounters the walls of the container and fills the entire available space.
- This is because the gas molecules are widely spaced, and the cohesive forces between them are very small. Unlike liquids, gases cannot form a free surface

What is a Fluid?

Difference between liquids and solids ?

Liquids	Gases
Difficult to compress and often regarded as incompressible	Easily to compress — changes of volume is large, cannot normally be neglected and are related to temperature
Occupies a fixed volume and will take the shape of the container	No fixed volume, it changes volume to expand to fill the containing vessels
A free surface is formed if the volume of container is greater than the liquid.	Completely fill the vessel so that no free surface is formed.

Application areas of Fluid Mechanics

- Mechanics of fluids is extremely important in many areasof engineering and science. Examples are:

Biomechanics

- Blood flow through arteries and veins
- Airflow in the lungs
- Flow of cerebral fluid

Households :

- Piping systems for cold water, natural gas, and sewage.
- Piping and ducting network of heating and air-conditioning systems
- refrigerator, vacuum cleaner, dish washer, washing machine, water meter, natural gas meter, air conditioner, radiator, etc.

Application areas of Fluid Mechanics

Mechanical Engineering

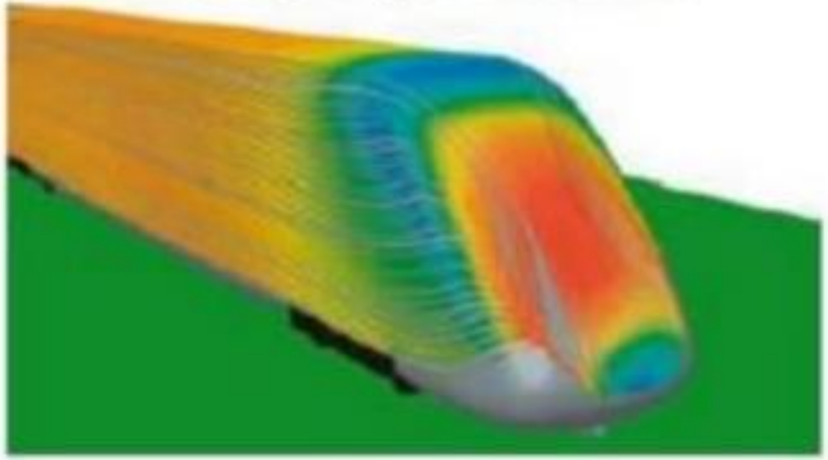
- Design of pumps, turbines, air-conditioning equipment, pollution-control equipment, etc.
- Design and analysis of aircraft, boats, submarines, rockets, jet engines, wind turbines, biomedical devices, the cooling of electronic components, and the transportation of water, crude oil, and natural gas.

Civil Engineering

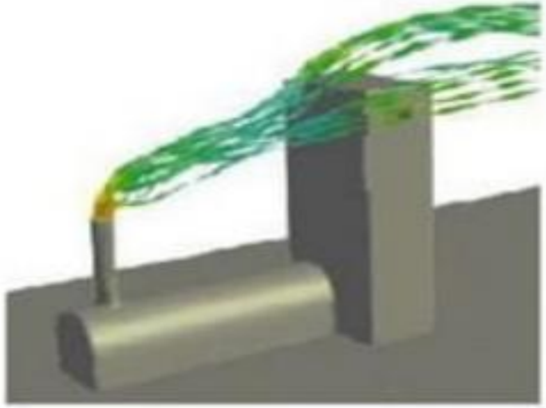
- Transport of river sediments.
- Pollution of air and water
- Design of piping systems
- Flood control systems

Application areas of Fluid Mechanics

High speed train

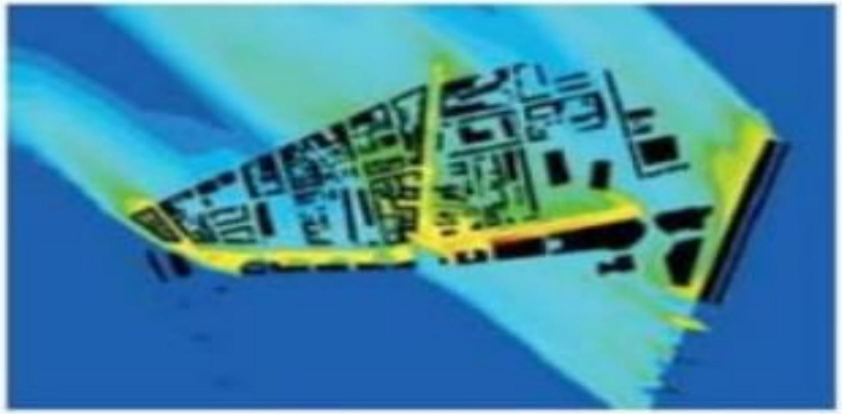


Wind turbines



Smoke from a stack

Pollutant dispersion over a city



Properties of Fluids

a)Density or Mass Density:

Density or mass density of a fluid is defined as the ratio of the mass of a fluid to its volume at specified temperature and pressure . Thus mass per unit volume of a fluid is called density. It is denoted the symbol **p (rho).**

The unit of mass density in SI unit : kg per cubic meter, i.e ., **kg/m**’.

- The density of liquids may be considered as constant while that of gases changes with the variation of pressure and temperature.

Mathematically mass density is written as. **Mass of fluid/Volume of fluid**

Density or Mass Density:

- The density of a substance, in general, depends on temperature and pressure.
- The density of most gases is proportional to pressure and inversely proportional to temperature.
- Liquids and solids, on the other hand, are essentially incompressible substances, and the variation of their density with pressure is usually negligible.

b) Specific weight or Weight Density :

- Specific weight or weight density of a fluid is the ratio between the weight of a fluid to its volume.

- Thus weight per unit volume of a fluid 1s called weight density and it is denoted by the symbol w.

$$w = \frac{\text{Weight of fluid}}{\text{Volume of fluid}} = \frac{(\text{Mass of fluid}) \text{ x Acceleration due to gravity}}{\text{Volume of fluid}}$$

$$= \frac{\text{Mass of fluid x g}}{\text{Volume of fluid}}$$

$$= \rho \text{ x g}$$

$$w = \rho g$$

c) Specific Volume :

- Specific volume of a fluid is defined as the volume of a fluid occupied by a unit mass or volume per unit mass of a fluid is called specific volume.

Mathematically, it is expressed as

$$\text{Specific volume} = \frac{\text{Volume of fluid}}{\text{Mass of fluid}} = \frac{1}{\frac{\text{Mass of fluid}}{\text{Volume}}} = \frac{1}{\rho}$$

- Thus specific volume is the reciprocal of mass density. It is expressed as **m3/kg**.
- It 1s commonly applied to gases.

Specific Gravity (S.G) :

- Specific gravity is defined as the ratio of the weight density (or density) of a fluid to the weight density (or density) of a standard fluid.

- For liquids, the standard fluid is taken water and for gases, the standard fluid is taken air.

Specific gravity is also called relative density. It is dimensionless quantity and is denoted by the symbol S.

$$S(\text{for liquids}) = \frac{\text{Weight density (density) of liquid}}{\text{Weight density (density) of water}}$$

$$S(\text{for gases}) = \frac{\text{Weight density (density) of gas}}{\text{Weight density (density) of air}}$$

$$\text{Thus weight density of a liquid} = S \times \text{Weight density of water}$$
$$= S \times 1000 \times 9.81 \text{ N/m}^3$$

$$\text{Thus density of a liquid} = S \times \text{Density of water}$$
$$= S \times 1000 \text{ kg/m}^3$$

Specific gravities of some substances at 0°C

Substance	SG
Water	1.0
Blood	1.05
Seawater	1.025
Gasoline	0.7
Ethyl alcohol	0.79
Mercury	13.6
Wood	0.3–0.9
Gold	19.2
Bones	1.7–2.0
Ice	0.92
Air (at 1 atm)	0.0013

Example 1.

Calculate the specific weight, density and specific gravity of one liter of a liquid which weighs 7 N.

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Solution. Given :

$$\text{Volume} = 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \quad \left(\because\ 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \text{ or } 1 \text{ litre} = 1000 \text{ cm}^3\right)$$

$$\text{Weight} = 7 \text{ N}$$

(*i*) Specific weight (*w*)

$$= \frac{\text{Weight}}{\text{Volume}} = \frac{7 \text{ N}}{\left(\frac{1}{1000}\right) \text{m}^3} = \mathbf{7000 \text{ N/m}^3. \text{ Ans.}}$$

(*ii*) Density (ρ)

$$= \frac{w}{g} = \frac{7000}{9.81} \text{ kg/m}^3 = \mathbf{713.5 \text{ kg/m}^3. \text{ Ans.}}$$

(*iii*) Specific gravity

$$= \frac{\text{Density of liquid}}{\text{Density of water}} = \frac{713.5}{1000} \quad \{\because\ \text{Density of water} = 1000 \text{ kg/m}^3\}$$

$$= \mathbf{0.7135. \text{ Ans.}}$$

Example 2. *Calculate the density, specific weight and weight of one liter of petrol of specific gravity = 0.7*

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Solution. Given : Volume = 1 litre = $1 \times 1000\ cm^3 = \frac{1000}{10^6}\ m^3 = 0.001\ m^3$

Sp. gravity $S = 0.7$

(*i*) *Density* (ρ)

Density (ρ) $= S \times 1000\ kg/m^3 = 0.7 \times 1000 = \mathbf{700\ kg/m^3}$. **Ans.**

(*ii*) *Specific weight* (*w*)

$$w = \rho \times g = 700 \times 9.81\ N/m^3 = \mathbf{6867\ N/m^3}. \textbf{ Ans.}$$

(*iii*) *Weight* (*W*)

We know that specific weight $= \frac{\text{Weight}}{\text{Volume}}$

or $$w = \frac{W}{0.001} \text{ or } 6867 = \frac{W}{0.001}$$

∴ $$W = 6867 \times 0.001 = \mathbf{6.867\ N}. \textbf{ Ans.}$$

Viscosity:

- Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid.

- When two layers of a fluid, a distance "dy" apart move one over the other at different velocities say **"u"** and **"u+ du"** as shown in Fig.1.1, the viscosity together with relative velocity causes a shear stress acting between the fluid layers:

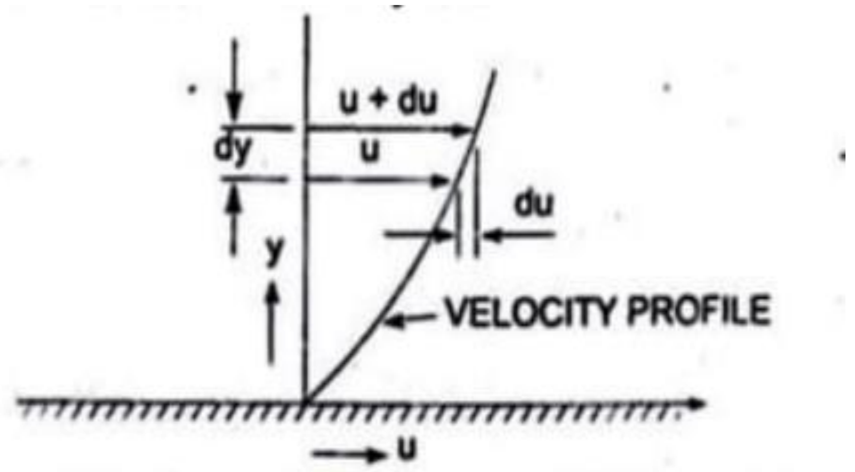
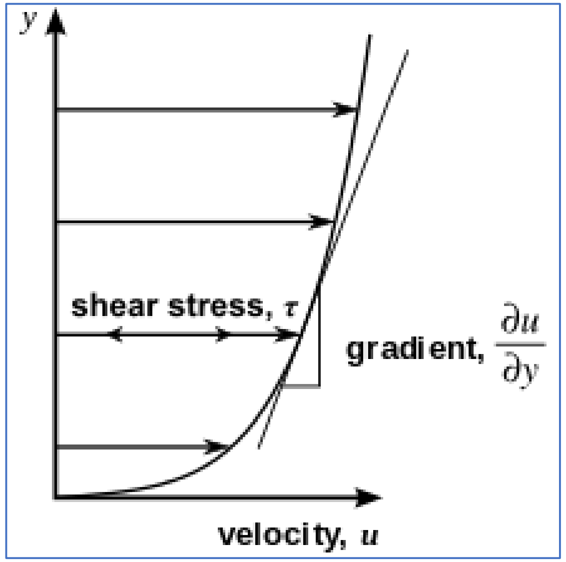


Fig. 1.1 Velocity variation near a solid boundary.

- The property of the fluid which offers internal resistance between two adjacent layers is known as viscosity
  a)cohesive forces
  b)molecular momentum transfer in perpendicular direction between two adjacent layers.


y
shear stress, τ
gradient, ∂u/∂y
velocity, u


Water
Olive Oil
Honey
Low viscosity
Medium viscosity
Medium strength intermolecular bonds
High viscosity

Viscosity:

- The top layer causes a shear stress on the adjacent lower layer while the **lower layer** causes a shear stress on the **adjacent top layer.**

- This shear stress is proportional to the rate of change of velocity with respect to y. It is denoted by symbol τ called **Tau.**

Mathematically,

$$\tau \propto \frac{du}{dy}$$

- where **μ** (called **mu**) is the constant of proportionality and is known as the coefficient of dynamic viscosity or only viscosity or Absolute viscosity

- du/dy represents the rate of shear strain

  or rate of shear deformation

  or velocity gradient.

$$\mu = \frac{\tau}{\frac{du}{dy}}$$

- Thus viscosity is also defined as the shear stress required to produce unit rate of shear strain.

- The unit of viscosity is obtained by putting the dimension of the quantities.

$$\mu = \frac{\text{Shear stress}}{\dfrac{\text{Change of velocity}}{\text{Change of distance}}} = \frac{\text{Force/Area}}{\left(\dfrac{\text{Length}}{\text{Time}}\right) \times \dfrac{1}{\text{Length}}}$$

$$= \frac{\text{Force/(length)}^2}{\dfrac{1}{\text{Time}}} = \frac{\text{Force} \times \text{Time}}{(\text{Length})^2}$$

$$\text{SI unit of viscosity} = \frac{\text{Newton second}}{\text{m}^2} = \frac{\text{Ns}}{\text{m}^2}$$

Kinematic Viscosity/absolute viscosity/dynamic viscosity :

- It is defined as the ratio between the dynamic viscosity and density of fluid.

lt is denoted by the Greek symbol (v) called "nu’ .

Mathematically ,

$$\nu = \frac{\text{Viscosity}}{\text{Density}} = \frac{\mu}{\rho}$$

- The SI unit of kinematic viscosity is m/s.

Newton's Law of Viscosity:

- It states that the shear stress (τ) on a fluid element layer in directly proportional to the rate of shear strain. The constant of proportionality is called the co-efficient viscosity. Mathematically, it is expressed as

$$\tau = \mu \frac{du}{dy}$$

- Fluids which obey the above relation are known as **Newtonian fluids** and the fluids which do not obey the above relation are called **Non-Newtonian fluids**.

Variation of Viscosity with Temperature

- Temperature affects the viscosity.

- The viscosity of liquids **decreases** with the **increase** of temperature while the viscosity of gases increases with increase of temperature.

  This is due to reason that the viscous forces in a fluid are due to **cohesive forces** and molecular momentum transfer.

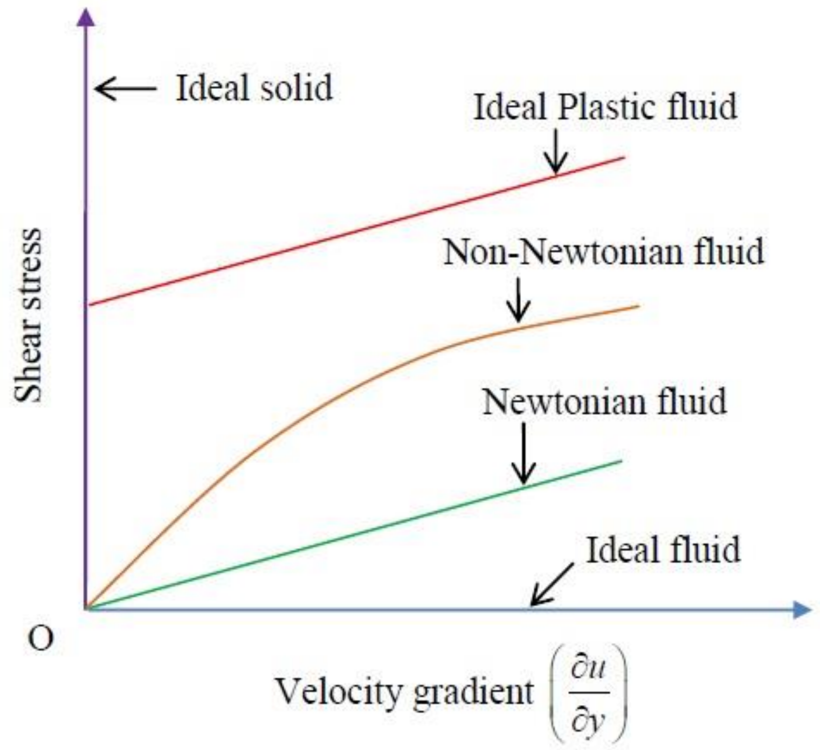
- In liquids the cohesive forces predominates the molecular momentum transfer due to closely packed molecules and with the increase in temperature, the cohesive forces decreases with the result of decreasing viscosity.

Types of Fluids

a)Ideal Fluid :A fluid, which is incompressible and is having no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity.

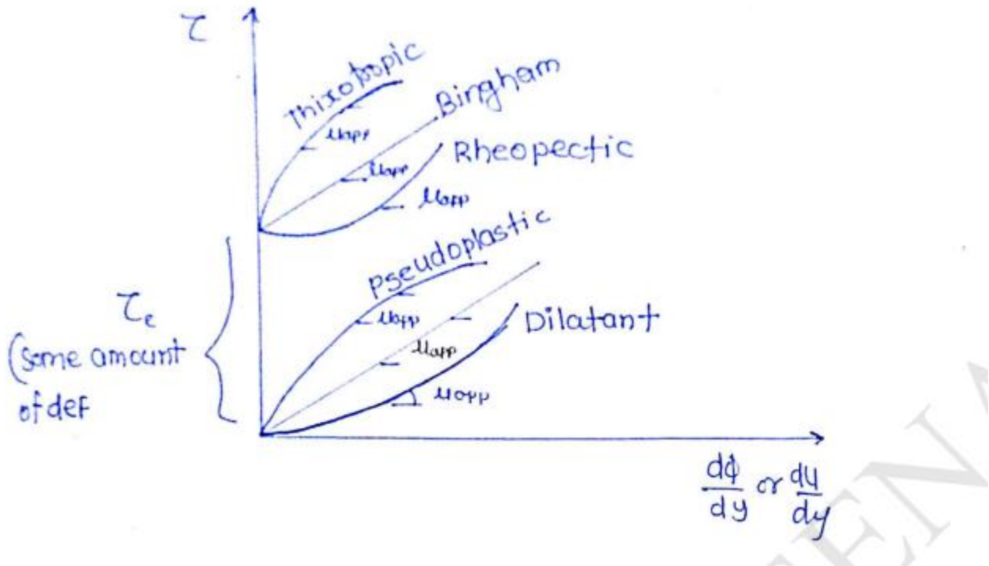
b)Real fluid : A fluid, which possesses viscosity, is known as real fluid. All the fluids: in actual practice, are real fluids.

C)Newtonian Fluid :A real fluid, in which the shear stress indirectly, proportional to the rate of shear strain (or **velocity gradient**), is known as a Newtonian fluid.



d) Non-Newtonian fluid : A real fluid, in which shear stress is not proportional to the rate of shear strain (or velocity gradient), known as a Non-Newtonian fluid.

e) Ideal Plastic Fluid. : A fluid, in which shear stress is more than the yield value and shear stress 1s proportional to the rate of shear strain (or velocity gradient), is known as ideal plastic fluid.



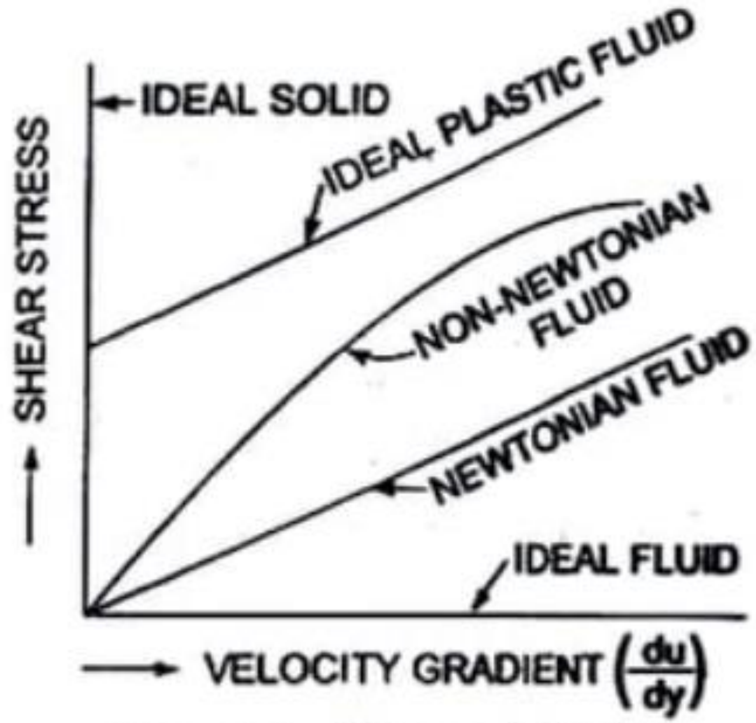


Fig. 1.2 *Types of fluids.*

Non- Linear $\tau = A(\frac{du}{dy})^n + \beta$

<table>
<tr><th colspan="3">Non-Newtonian Fluids $(\tau \neq \mu \frac{du}{dy})$</th></tr>
<tr><th colspan="2">Purely Viscous Fluids</th><th>Visco-elastic Fluids</th></tr>
<tr><th>Time - Independent</th><th>Time - Dependent</th><td rowspan="2">Visco- elastic Fluids
$\tau = \mu \frac{du}{dy} + \alpha E$
Example: Liquid-solid combinations in pipe flow.</td></tr>
<tr><td>1. Pseudo plastic Fluids
$\tau = \mu\left(\frac{du}{dy}\right)^n ; n < 1$
Example: Blood, milk
2. Dilatant Fluids
$\tau = \mu\left(\frac{du}{dy}\right)^n ; n > 1$
Example: Butter
3. Bingham or Ideal Plastic Fluid
$\tau = \tau_o + \mu\left(\frac{du}{dy}\right)^n$
Example: Water suspensions of clay and flyash</td><td>1.Thixotropic Fluids
$\tau = \mu\left(\frac{du}{dy}\right)^n + f(t)$
$f(t)$ *is decreasing*
Example: Printer ink; crude oil
2. Rheopectic Fluids
$\tau = \mu\left(\frac{du}{dy}\right)^n + f(t)$
$f(t)$ *is increasing*
Example: Rare liquid solid suspension</td></tr>
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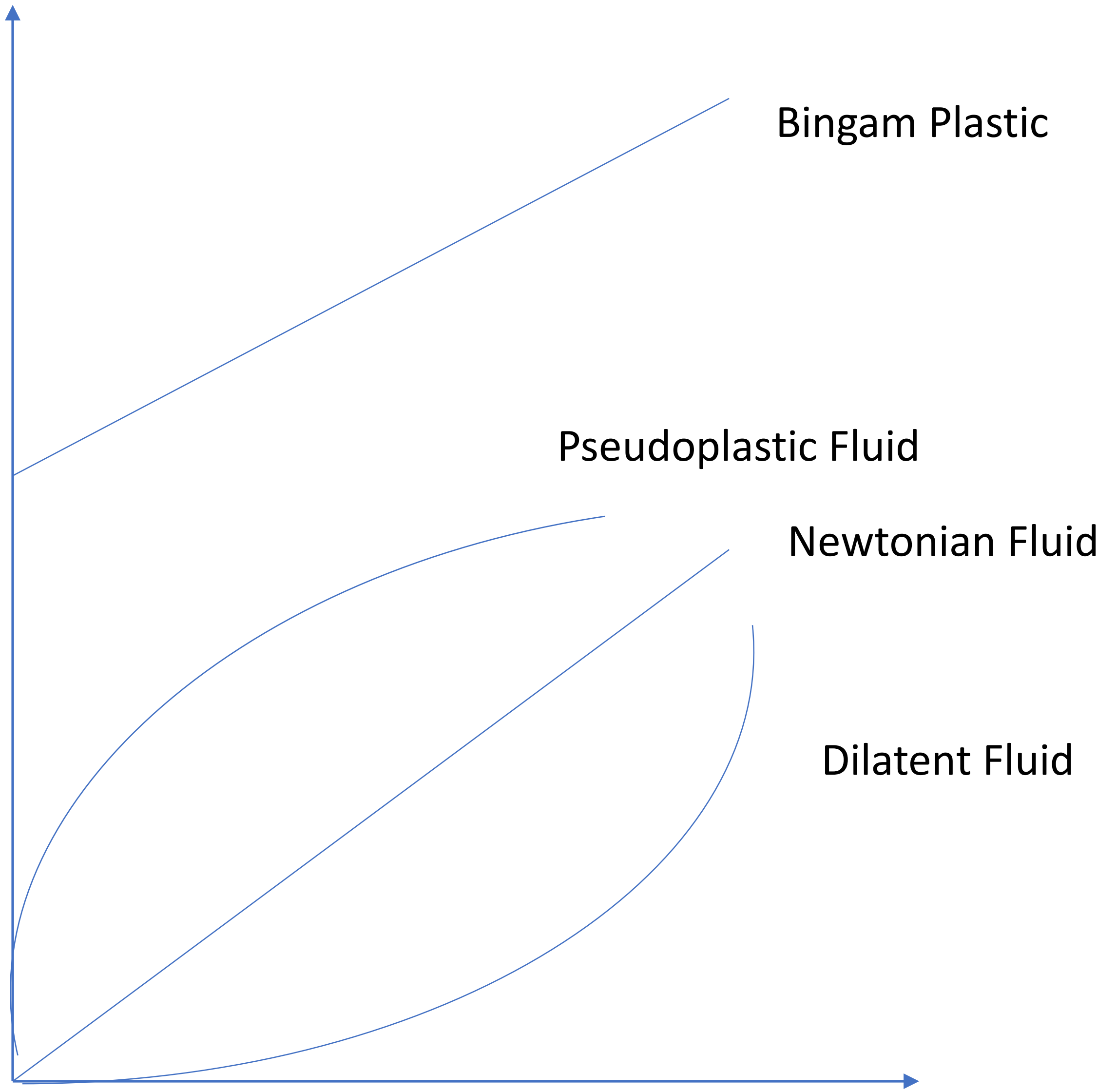
Non-Newtonian Fluids

Non- Linear $\tau = A(\frac{du}{dy})^n + \beta$

A) μ varies with $\frac{d\theta}{dt}$



Example 3

If the velocity distribution over a plate is given by

$$u = \frac{2}{3}y - y^2$$

in which u is velocity in metre per second at a distance y metre above the plate, determine the shear stress at $y = 0$ *and y= 0.15 m. Take dynamic viscosity of fluid as 8.63 poises.*

Solution. Given : $u = \frac{2}{3}y - y^2 \quad \therefore \quad \frac{du}{dy} = \frac{2}{3} - 2y$

$$\left(\frac{du}{dy}\right)_{at\ y=0} \text{ or } \left(\frac{du}{dy}\right)_{y=0} = \frac{2}{3} - 2(0) = \frac{2}{3} = 0.667$$

Also $$\left(\frac{du}{dy}\right)_{at\ y=0.15} \text{ or } \left(\frac{du}{dy}\right)_{y=0.15} = \frac{2}{3} - 2 \times .15 = .667 - .30 = 0.367$$

Value of $\mu = 8.63$ poise $= \frac{8.63}{10}$ SI units $= 0.863$ N s/m^2

Now shear stress is given by equation (1.2) as $\tau = \mu \frac{du}{dy}$.

(*i*) Shear stress at $y = 0$ is given by

$$\tau_0 = \mu \left(\frac{du}{dy}\right)_{y=0} = 0.863 \times 0.667 = \mathbf{0.5756\ N/m^2}. \textbf{ Ans.}$$

(*ii*) Shear stress at $y = 0.15$ m is given by

$$(\tau)_{y=0.15} = \mu \left(\frac{du}{dy}\right)_{y=0.15} = 0.863 \times 0.367 = \mathbf{0.3167\ N/m^2}. \textbf{ Ans.}$$

Problem 1.4 *A plate 0.025 mm distant from a fixed plate, moves at 60 cm/s and requires a force of 2 N per unit area i.e., 2 N/m² to maintain this speed. Determine the fluid viscosity between the plates.*

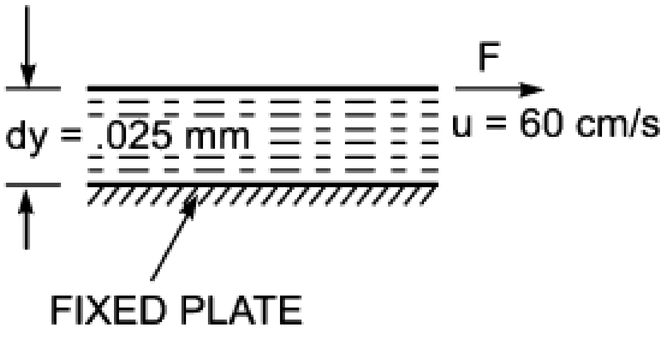


Fig. 1.3

Problem 1.4 *A plate 0.025 mm distant from a fixed plate, moves at 60 cm/s and requires a force of 2 N per unit area i.e., 2 N/m² to maintain this speed. Determine the fluid viscosity between the plates.*

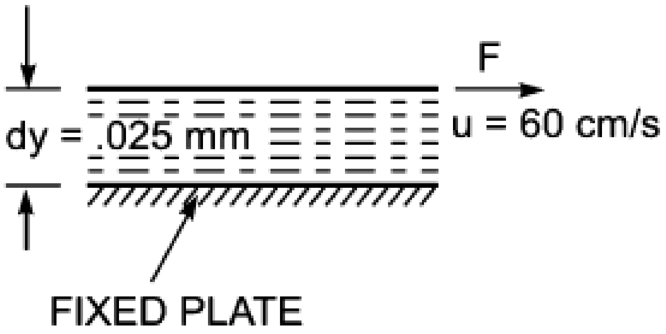


Fig. 1.3

Distance between plates, $dy = .025$ mm

$= .025 \times 10^{-3}$ m

Velocity of upper plate, $u = 60$ cm/s $= 0.6$ m/s

Force on upper plate, $F = 2.0 \dfrac{\text{N}}{\text{m}^2}$.

Using the equation (1.2), we have $\tau = \mu \dfrac{du}{dy}$.

where du = Change of velocity = $u - 0 = u = 0.60$ m/s

dy = Change of distance = $.025 \times 10^{-3}$ m

τ = Force per unit area = $2.0 \dfrac{\text{N}}{\text{m}^2}$

$$2.0 = \mu \frac{0.60}{.025 \times 10^{-3}}$$

$$\therefore \quad \mu = \frac{2.0 \times .025 \times 10^{-3}}{0.60} = 8.33 \times 10^{-5} \frac{\text{Ns}}{\text{m}^2}$$

$= 8.33 \times 10^{-5} \times 10$ poise $=$ **8.33×10^{-4} poise. Ans.**

Thermodynamic Properties

- Fluids consist of liquids or gases. But gases are compressible fluids and hence thermodynamic properties play an important role.

- With the change of pressure and temperature, the gases undergo large variation in density.

- The relationship between pressure (absolute), specific volume and temperature (absolute) of a gas is given by the equation of state as

$$PV = mRT$$

where P = Absolute pressure of a gas in N/m”

V= Volume

m = Mass

R = Gas Constant

T = Absolute Temperature

Thermodynamic Properties:

- The value of gas constant R is R =287 **Unit** :J/kg K

Isothermal Process: If the changes in density occurs at constant temperature, then the process is called isothermal and relationship between pressure (p) and density (p) is given by

$$\frac{p}{\rho} = \text{constant}$$

Adiabatic Process. If the change in density occurs with no heat exchange to and from the gas, the process is called adiabatic. And if no heat is generated within the gas due to friction, the relationship between pressure and density is given by

$$\frac{p}{\rho^k} = \text{constant}$$

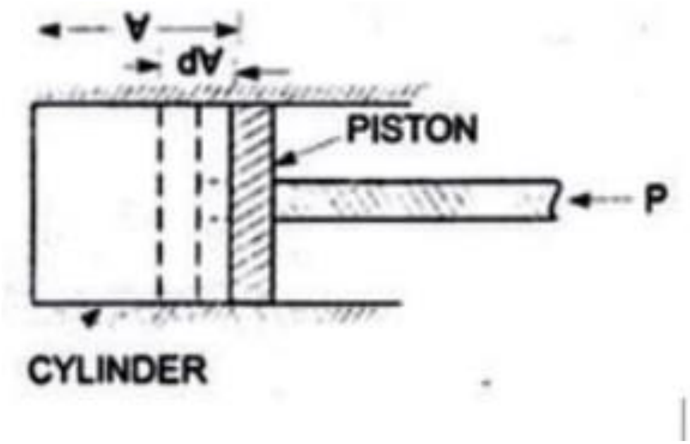
Thermodynamic Properties

where k = Ratio of specific heat of a gas at constant pressure and constant volume.

k = 1.4 for air

Compressibility (β) and Bulk Modulus (K)

- Compressibility (β) is the reciprocal of the bulk modulus of elasticity, K which is defined as the ratio of compressive stress to volumetric strain.



Let V= Volume of a gas enclosed in the cylinder

P =Pressure of gas when volume is V

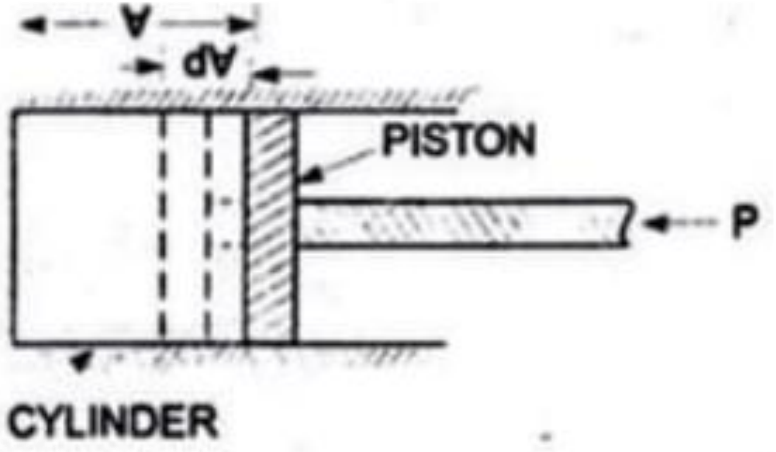
Let the pressure is increased fo P+ dp,

the volume of gas decreases from V to V— dV.

Then increase in pressure = dp
Decrease in volume =dV
Volumetric strain = - dV/V

- **- ve** sign means the volume Decreases with increase of Pressure.

$$\therefore \text{Bulk modules} \quad K = \frac{\text{Increase of pressure}}{\text{Volumetric strain}} = \frac{dp}{\frac{-dV}{V}} = -\frac{dp}{dV}V$$



- Compressibility is given by = 1/K

Surface Tension and Capillarity

Surface tension : is defined as the tensile force acting on the surface of a liquid in contact with a gas or on the surface between two immiscible liquids such that the contact surface behaves like a membrane under tension.

- Surface tension is created due to the unbalanced cohesive forces acting on the liquid molecules at the fluid surface.

- Molecules in the interior of the fluid mass are surrounded by molecules that are attracted to each other equally.

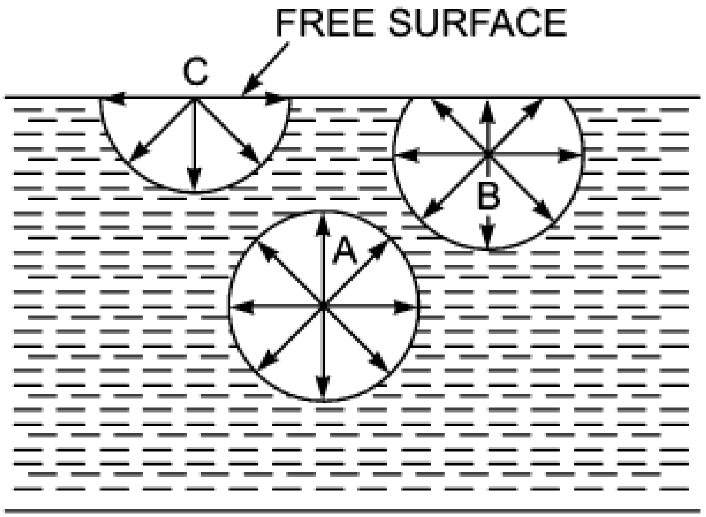
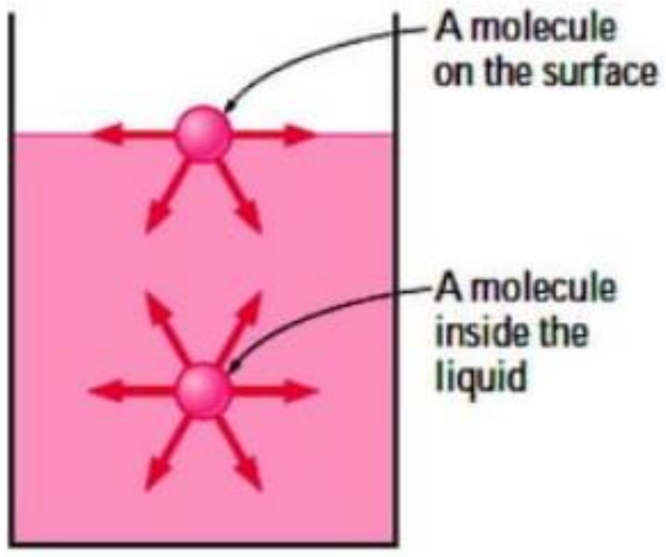


Fig. 1.10 *Surface tension.*

- However, molecules along the surface are subjected to a net force toward the interior.

- The apparent physical consequence of this unbalanced force along the surface is to create the hypothetical skin or membrane.

- A tensile force may be considered to be acting in the plane of the surface along any line in the surface.
- The intensity of the molecular attraction per unit length along any line in the surface is a called the surface tension. inside the liquid



- It 1s denoted by Greek letter σ (called sigma).
- The SI unit is N/m.

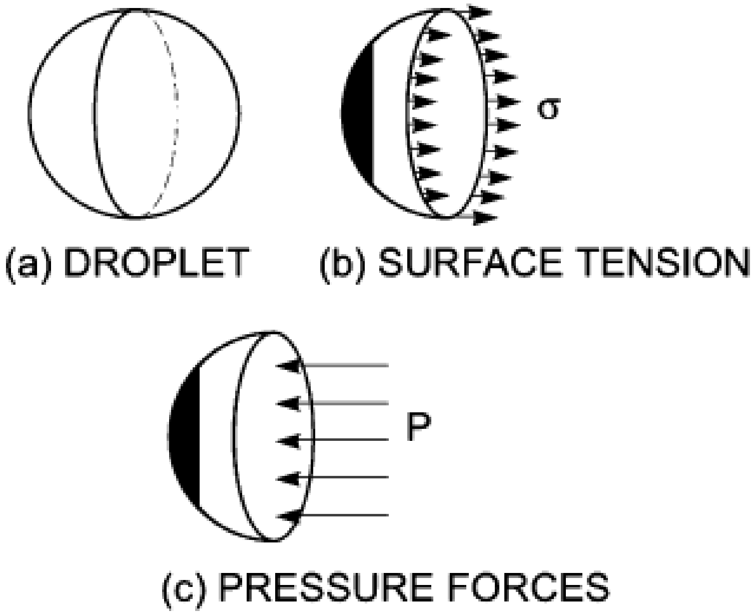
Surface Tension on Liquid Droplet :

Consider a small spherical droplet of a liquid of radius 'r'. On the entire surface of the droplet, the tensile force due to surface tension will be acting.

Let σ Surface tension of the liquid

p = Pressure intensity inside the droplet (in excess of the outside pressure intensity)

d = Dia. of droplet.



(*i*) tensile force due to surface tension acting around the circumference of the cut portion as shown in Fig. 1.11 (*b*) and this is equal to

$$= \sigma \times \text{Circumference}$$

$$= \sigma \times \pi d$$

(*ii*) pressure force on the area $\frac{\pi}{4} d^2 = p \times \frac{\pi}{4} d^2$ as shown in

Fig. 1.11 (*c*). These two forces will be equal and opposite under equilibrium conditions, *i.e.*,

$$p \times \frac{\pi}{4} d^2 = \sigma \times \pi d$$

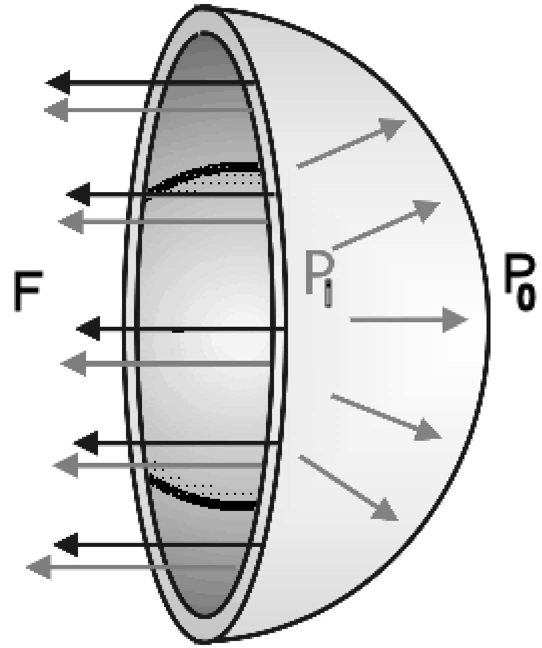
or

$$p = \frac{\sigma \times \pi d}{\frac{\pi}{4} \times d^2} = \frac{4\sigma}{d} \qquad \text{...(1.14)}$$

1.6.2 Surface Tension on a Hollow Bubble. A hollow bubble like a soap bubble in air has two surfaces in contact with air, one inside and other outside. Thus two surfaces are subjected to surface tension. In such case, we have

$$p \times \frac{\pi}{4} d^2 = 2 \times (\sigma \times \pi d)$$

$$\therefore \qquad p = \frac{2\sigma\pi d}{\frac{\pi}{4} d^2} = \frac{8\sigma}{d} \qquad \text{...(1.15)}$$



1.6.3 Surface Tension on a Liquid Jet. Consider a liquid jet of diameter '*d*' and length 'L' as shown in Fig. 1.12.

Let p = Pressure intensity inside the liquid jet above the outside pressure

σ = Surface tension of the liquid.

Consider the equilibrium of the semi jet, we have

Force due to pressure $= p \times$ area of semi jet

$= p \times L \times d$

Force due to surface tension $= \sigma \times 2L.$

Equating the forces, we have

$$p \times L \times d = \sigma \times 2L$$

$$\therefore \quad p = \frac{\sigma \times 2L}{L \times d} \quad \text{...(1.16)}$$

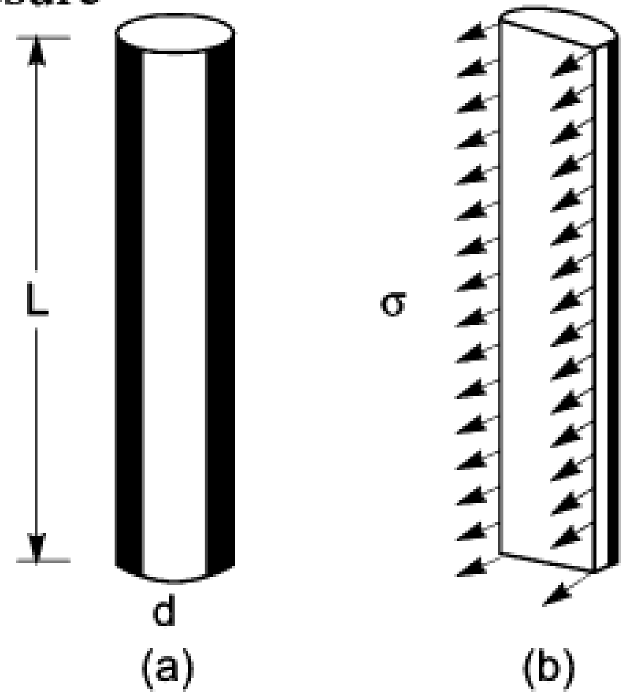


Fig. 1.12 *Forces on liquid jet.*

Problem 1.26 *Find the surface tension in a soap bubble of 40 mm diameter when the inside pressure is 2.5 N/m² above atmospheric pressure.*

Solution. Given :

Dia. of bubble, $d = 40 \text{ mm} = 40 \times 10^{-3} \text{ m}$

Pressure in excess of outside, $p = 2.5 \text{ N/m}^2$

For a soap bubble, using equation (1.15), we get

$$p = \frac{8\sigma}{d} \quad \text{or} \quad 2.5 = \frac{8 \times \sigma}{40 \times 10^{-3}}$$

$$\sigma = \frac{2.5 \times 40 \times 10^{-3}}{8} \text{ N/m} = \mathbf{0.0125 \text{ N/m. Ans.}}$$

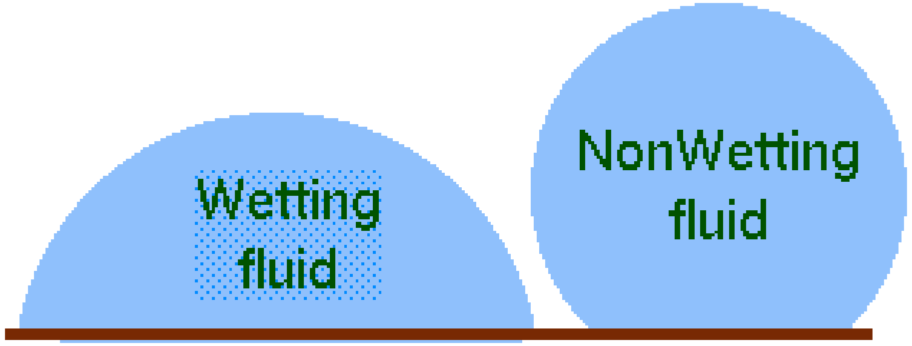
Surface Tension and Capillarity

- Capillarity is defined as a phenomenon of rise or fall of a liquid surface in a small tube relative to the adjacent general level of liquid when the tube is held vertically in the liquid.

- The **rise** of liquid surface is known as **capillary rise** while the **fall** of the liquid surface is known as **capillary depression.**

- The attraction (adhesion) between the wall of the tube and liquid molecules is strong enough to overcome the mutual attraction (cohesion) of the molecules and pull them up the wall. Hence, the liquid is said to wet the solid surface.

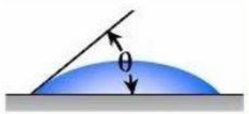
- It is expressed in terms of cm or mm of liquid. Its value depends upon the specific weight of the liquid, diameter of the tube and surface tension of the liquid.



Wetting and non-wetting liquid

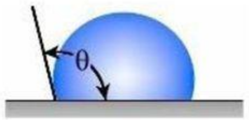
- Wetting Liquid

$$\theta < \frac{\pi}{2}$$

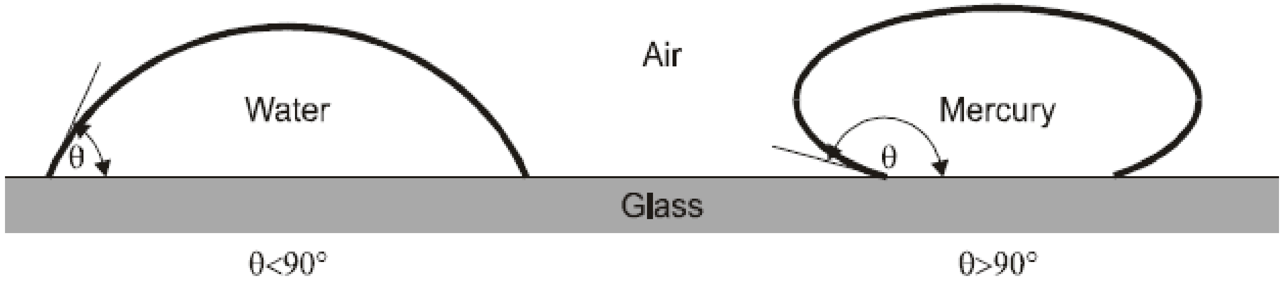


- Non-wetting Liquid

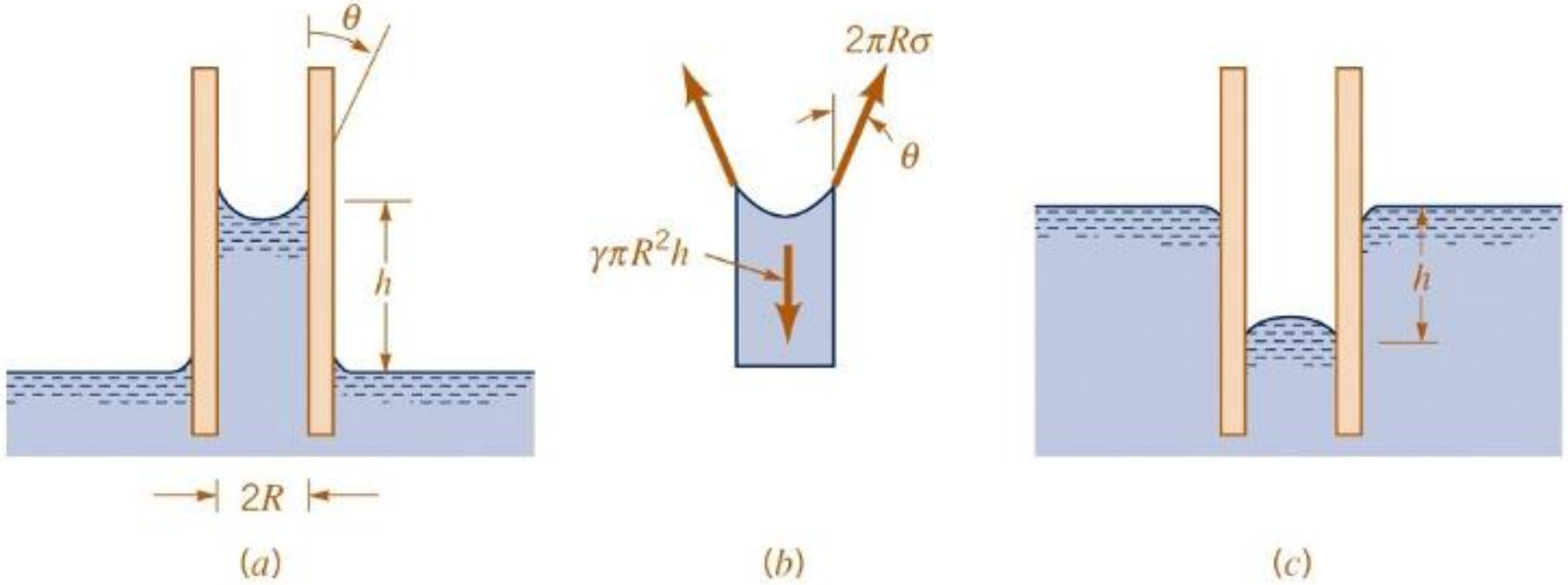
$$\theta > \frac{\pi}{2}$$



- For pure water in contact with a clean glass surface θ is essentially zero degree.



Capillary Action



Wetting Liquid

Non- Wetting Liquid

Angle of contact for non wetting liquids is more than 90°

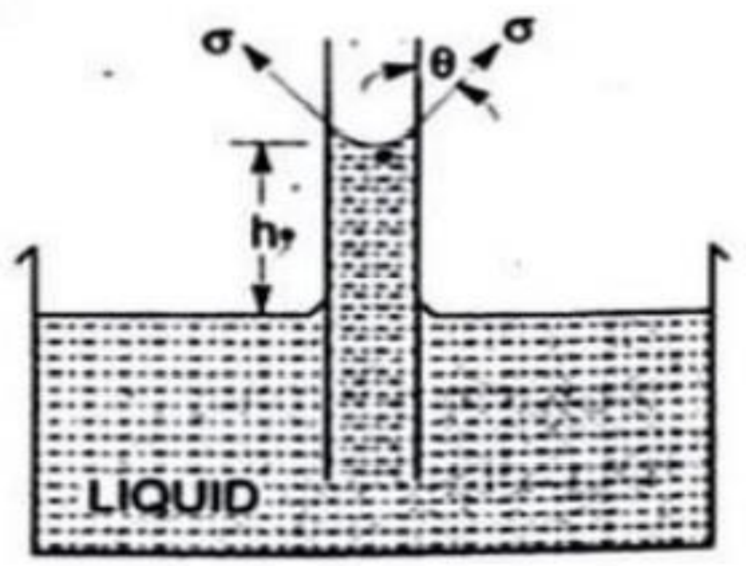
$$h = \frac{2\sigma cos\theta}{\gamma R}$$

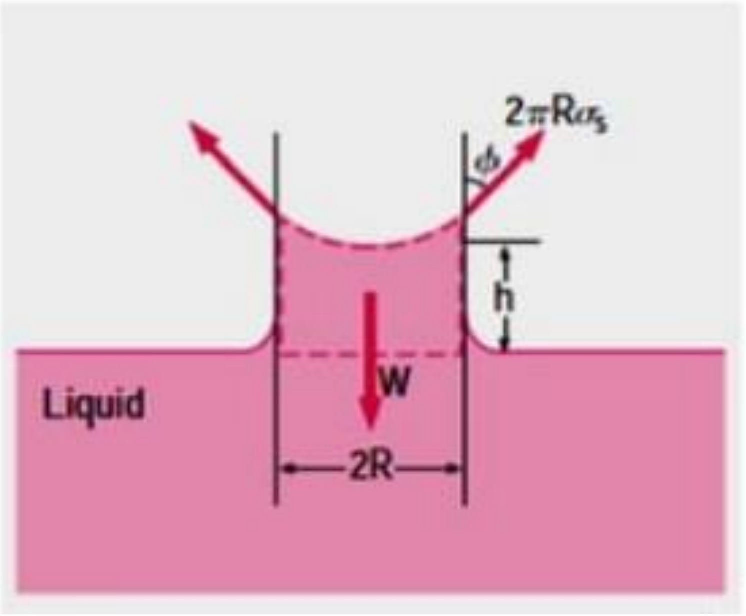
γ is specific weight; weight per unit volume

Surface Tension and Capillarity

Expression for Capillary Rise

- Consider a glass tube of small diameter 'd' opened at both ends and is inserted in a liquid, say water.
- The liquid will rise in the tube above the level of the liquid.
- Let h = the height of the liquid in the tube . Under a state of equilibrium, the weight of the liquid of height h is balanced by the force at the surface of the liquid in the tube. But the force at the surface of the liquid in the tube is due to surface tension.





Expression for Capillary Rise

Let Sigma = Surface tension of liquid

teta(θ) = Angle of contact between the liquid and glass tube

The weight of the liquid of height h in the tube =

(Area of the tube x h)x p x g

$$= \frac{\pi}{4} d^2 \times h \times \rho \times g$$

where ρ = Density of liquid

Vertical component of the surface tensile force

$$= (\sigma \times \text{Circumference}) \times \cos\theta$$

$$= \sigma \times \pi d \times \cos\theta$$

For equilibrium, equating (1.17) and (1.18), we get

$$\frac{\pi}{4} d^2 \times h \times \rho \times g = \sigma \times \pi d \times \cos\theta$$

or

$$h = \frac{\sigma \times \pi d \times \cos\theta}{\frac{\pi}{4} d^2 \times \rho \times g} = \frac{4\sigma\cos\theta}{\rho \times g \times d}$$

Expression for Capillary Rise

- The value of teta between water and clean glass tube is approximately equal to zero and hence cos (teta) is equal to unity. Then rise of water 1s given by

$$h = \frac{4\sigma}{\rho \times g \times d}$$

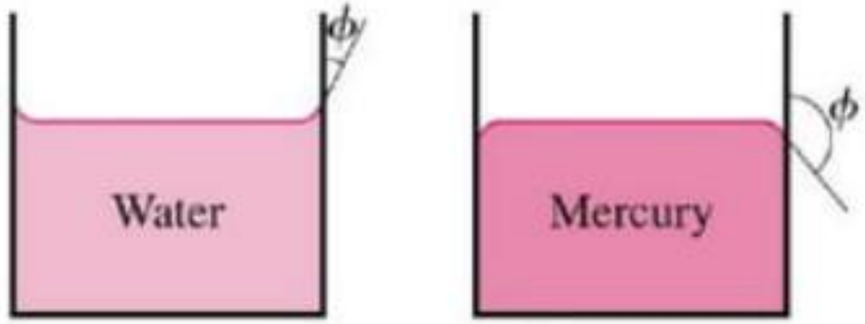
- Contact angle depends on both the liquid and the solid.

- If teta is less than 90°, the liquid is said to "wet" the solid.

  However, if teta is greater than 90°, the liquid is repelled by the solid, and tries not to "wet“ – (Non wetting) it.

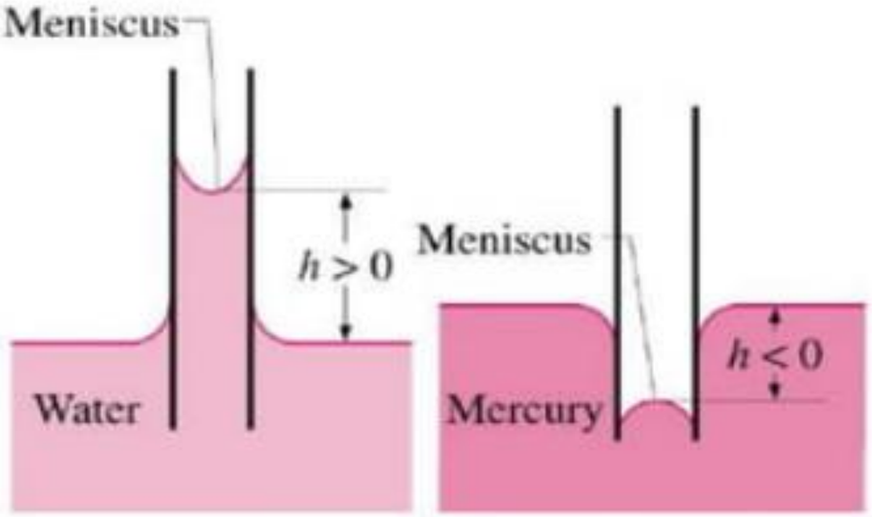
- For example, water wets glass, but not wax. Mercury on the other hand does not wet glass.

Capillarity :



(*a*) Wetting fluid

(*b*) Nonwetting fluid



If the glass tube is dipped in mercury, the revel of mercury in the tube will be lower than the general level of the outside liquid as shown above.

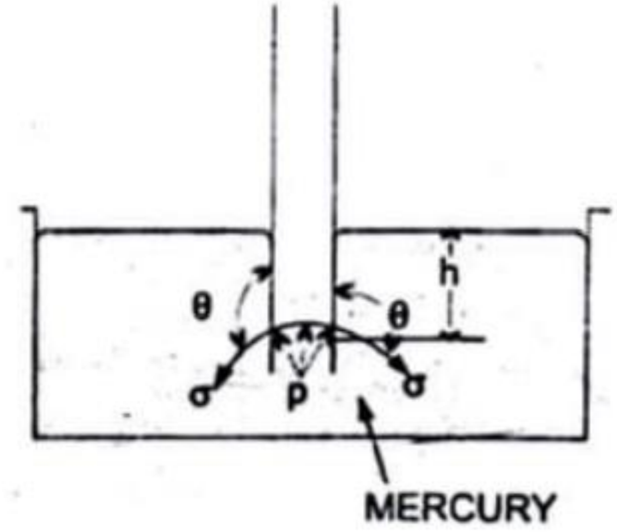
Expression for Capillary Fall

Let h = Height of depression in tube.

- Then in equilibrium, two forces arc acting on the mercury inside the tube.

- First one is due to surface tension acting in the downward direction and is equal to $\sigma \times \pi d \times \cos\theta$.

- Second force is due to hydrostatic force acting upward and is equal to intensity of pressure at a depth 'h' x Area



Expression for Capillary Fall

$$= p \times \frac{\pi}{4} d^2 = \rho g \times h \times \frac{\pi}{4} d^2 \ \{\because p = \rho g h\}$$

Equating the two, we get

$$\sigma \times \pi d \times \cos\theta = \rho g h \times \frac{\pi}{4} d^2$$

$$h = \frac{4\sigma\cos\theta}{\rho g d}$$

Value of θ for mercury and glass tube is 128°

Capillarity…Example 1

- Calculate the capillary rise in a glass tube of 2.5 mm diameter when immersed vertically in (a) water and (b) mercury. Take surface tensions $\sigma = 0.0725$ N/m for water and $\sigma = 0.52$ N/m for mercury in contact with air. The specific gravity for mercury is given as 13.6 and angle of contact = 130^{o}.

Capillarity...Example 1

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Solution. Given :

Dia. of tube,	$d = 2.5 \text{ mm} = 2.5 \times 10^{-3}$ m
Surface tenstion, σ for water	= 0.0725 N/m
σ for mercury	= 0.52 N/m
Sp. gr. of mercury	= 13.6

Capillarity…Example 1

$\therefore$ Density $= 13.6 \times 1000 \text{ kg/m}^3$.

(*a*) **Capillary rise for water ($\theta = 0$)**

Using equation (1.20), we get $h = \dfrac{4\sigma}{\rho \times g \times d} = \dfrac{4 \times 0.0725}{1000 \times 9.81 \times 2.5 \times 10^{-3}}$

$= .0118$ m $=$ **1.18 cm. Ans.**

(*b*) **For mercury**

Angle of constant between mercury and glass tube, $\theta = 130°$

Using equation (1.21), we get $h = \dfrac{4\sigma \cos\theta}{\rho \times g \times d} = \dfrac{4 \times 0.52 \times \cos 130°}{13.6 \times 1000 \times 9.81 \times 2.5 \times 10^{-3}}$

$= -.004$ m $=$ **– 0.4 cm. Ans.**

The negative sign indicates the capillary depression.

Capillarity…Example 2

- Find out the minimum size of glass tube that can be used to measure water level if the capillary rise in the tube is to be restricted to 2 mm. Consider surface tension of water in contact with air as 0.073575 N/m.

- Find out the minimum size of glass tube that can be used to measure water level if the capillary rise in the tube is to be restricted to 2 mm. Consider surface tension of water in contact with air as 0.073575 N/m.

Solution. Given :

Capillary rise, $h = 2.0 \text{ mm} = 2.0 \times 10^{-3}$ m

Surface tension, $\sigma = 0.073575$ N/m

Let dia. of tube $= d$

The angle θ for water $= 0$

The density for water, $\rho = 1000 \text{ kg/m}^3$

$$h = \frac{4\sigma}{\rho \times g \times d} \text{ or } 2.0 \times 10^{-3} = \frac{4 \times 0.073575}{1000 \times 9.81 \times d}$$

$$\therefore \quad d = \frac{4 \times 0.073575}{1000 \times 9.81 \times 2 \times 10^{-3}} = 0.015 \text{ m} = \mathbf{1.5 \text{ cm}}. \textbf{ Ans.}$$

Thus minimum diameter of the tube should be 1.5 cm.

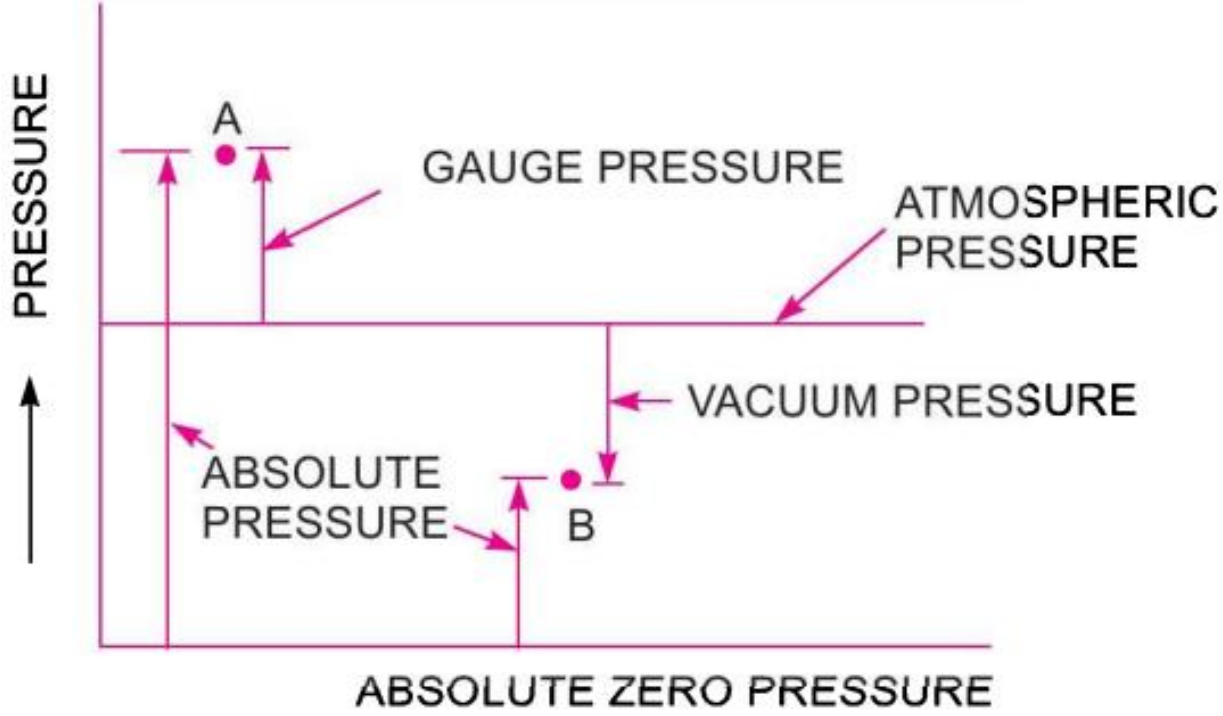
Pressure Measurement

1. Atmospheric Pressure

2. Absolute Pressure

3.Gauge Pressure

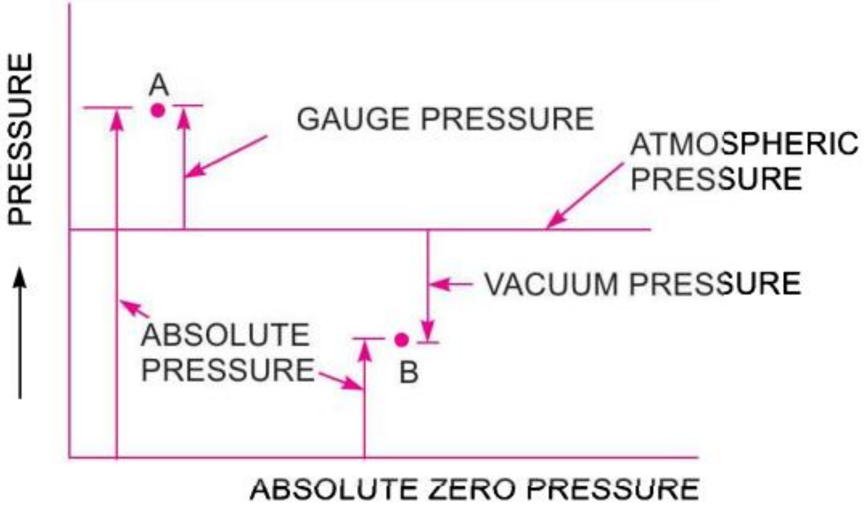
4.Vaccum   Pressure



1.Atmospheric Pressure : The Pressure is exerted by the Environmental mass on the earth surface

P_{atm} = 101.325 K pa

2. Absolute pressure : Absolute pressure is defined as the pressure which is measured with reference to absolute vacuum pressure.



2. Gauge pressure is defined as the pressure which is measured with the help of a pressure measuring instrument, in which the atmospheric pressure is taken as datum.

The atmospheric pressure on the scale is marked as zero.

3. Vacuum pressure is defined as the pressure below the atmospheric pressure.

Mathematically :

(*i*) Absolute pressure

= Atmospheric pressure + Gauge pressure

or $$p_{ab} = p_{atm} + p_{gauge}$$

(*ii*) Vacuum pressure

= Atmospheric pressure – Absolute pressure.

Atmospheric Pressure : The Pressure is exerted by the Environmental mass on the earth surface

$$P_{atm} = 101.325 \text{ K pa}$$

Note. (*i*) The atmospheric pressure at sea level at 15°C is 101.3 kN/m^2 or 10.13 N/cm^2 in SI unit. In case of MKS units, it is equal to 1.033 kgf/cm^2.

(*ii*) The atmospheric pressure head is 760 mm of mercury or 10.33 m of water.

Measurement of Pressure

The pressure of a fluid is measured by the following devices :

1. Manometers
2. Mechanical Gauges.

1.Manometers: Manometers are defined as the devices used for measuring the pressure at a point in a fluid by balancing the column of fluid by the same or another column of the fluid.

They are classified as :(a) Simple Manometers,

(b) Differential Manometers.

SIMPLE MANOMETERS : A simple manometer consists of a glass tube having one of its ends connected to a point where pressure is to be measured and other end remains open to atmosphere. Common types of simple manometers are :

1. Piezometer,
2. U-tube Manometer, and
3. Single Column Manometer.

1. Piezometer : It is the simplest form of manometer used for measuring gauge pressures. One end of this manometer is connected to the point where pressure is to be measured and other end is open to the atmosphere as shown in Figure.

The rise of liquid gives the pressure head at that point. If at a point A, the height of liquid say water is h in piezometer tube, then pressure at A

$$= \rho \times g \times h \ \frac{N}{m^2}.$$

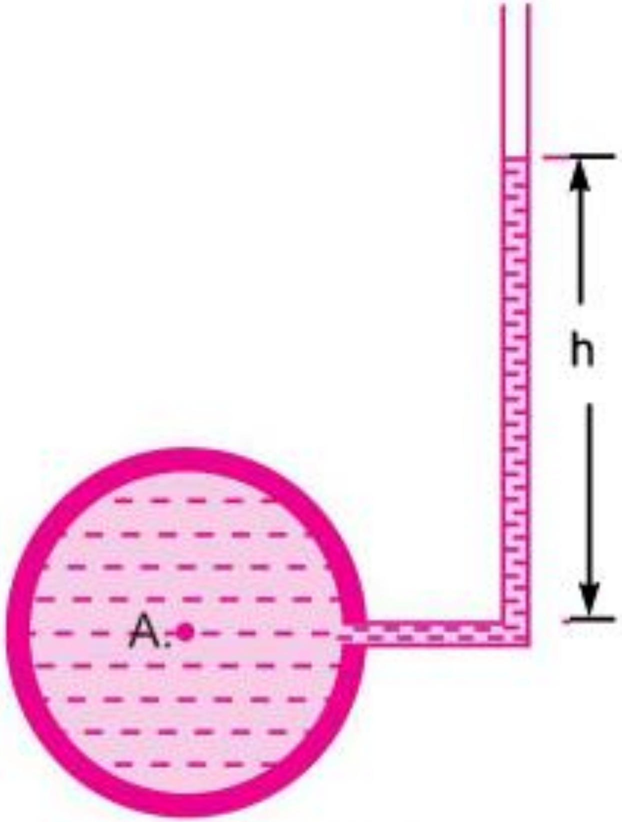
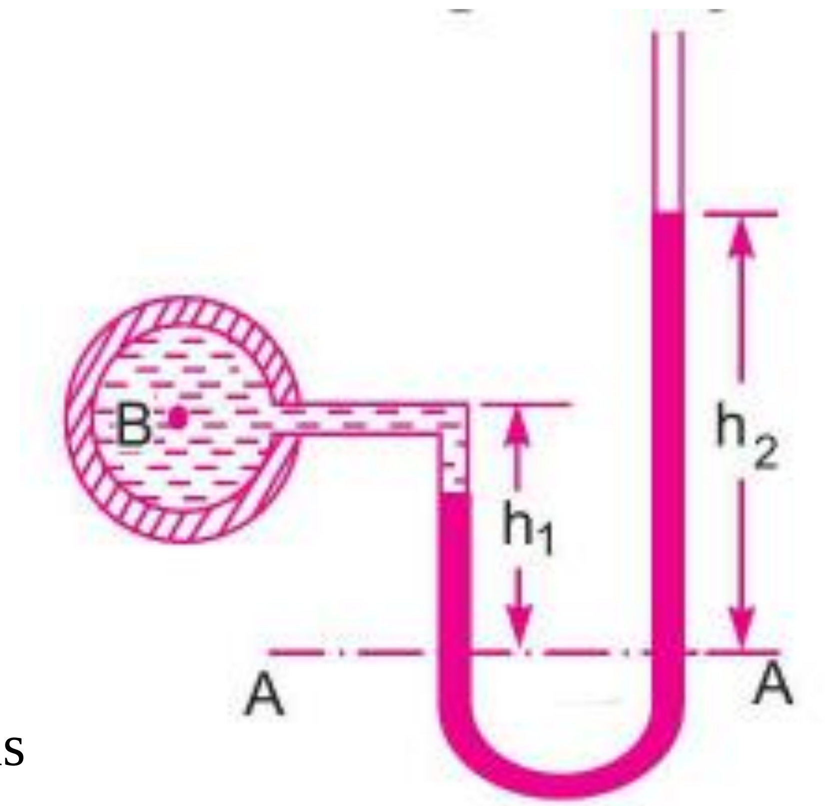


Fig. 2.8 *Piezometer.*

2.U-tube Manometer. It consists of glass tube bent in U-shape, one end of which is connected to a point at which pressure is to be measured and other end remains open to the atmosphere as shown in Figure. The tube generally contains mercury or any other liquid **whose specific gravity is greater than the specific gravity of the liquid whose pressure is to be measured.**



(a) For gauge pressure

(a) For Gauge Pressure.

Let B is the point at which pressure is to be measured, whose value is P . The datum line is A-A.

Let h_1 = Height of light liquid above the datum line

h_2 = Height of heavy liquid above the datum line

S_1 =p. gr. of light liquid

ρ_1= Density of light liquid = 1000 x S_1

S_2 = Sp. gr. of heavy liquid

ρ_1 = Density of heavy liquid = 1000 x S_2

(a) For Gauge Pressure.

Let B is the point at which pressure is to be measured, whose value is P . The datum line is A-A.

Let h_1 = Height of light liquid above the datum line

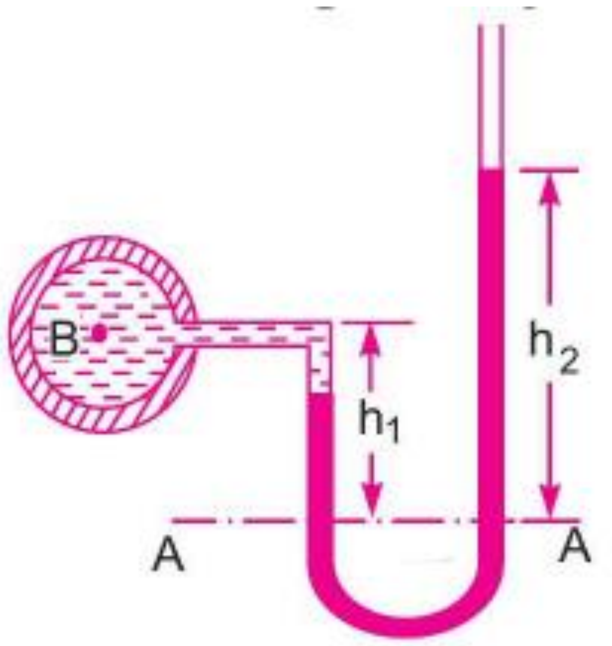
h_2 = Height of heavy liquid above the datum line

S_1 =p. gr. of light liquid

ρ_1= Density of light liquid = 1000 x S_1

S_2 = Sp. gr. of heavy liquid

ρ_1 = Density of heavy liquid = 1000 x S_2



(a) For gauge pressure

As the pressure is the same for the horizontal surface. Hence pressure above the horizontal datum line *A-A* in the left column and in the right column of U-tube manometer should be same.

Pressure above *A-A* in the left column $= p + \rho_1 \times g \times h_1$

Pressure above *A-A* in the right column $= \rho_2 \times g \times h_2$

Hence equating the two pressures $p + \rho_1 g h_1 = \rho_2 g h_2$

$\therefore$ $p = (\rho_2 g h_2 - \rho_1 \times g \times h_1)$.

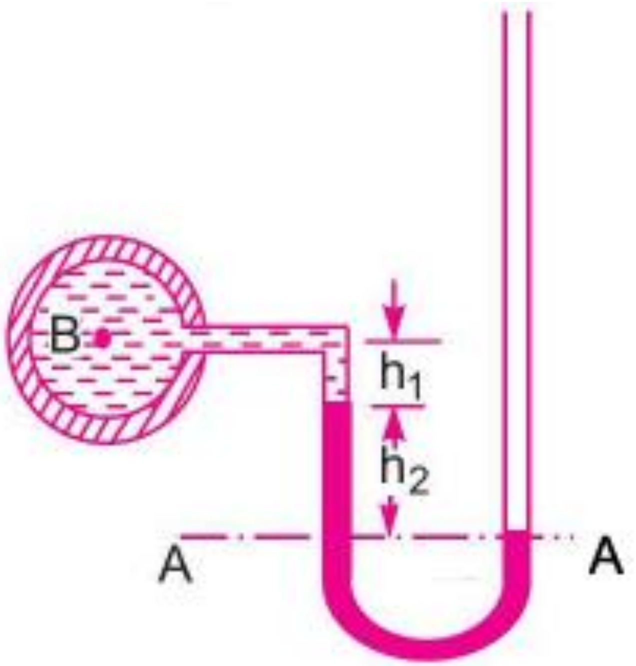
(*b*) **For Vacuum Pressure.** For measuring vacuum pressure, the level of the heavy liquid in the manometer will be as shown in Fig. 2.9 (*b*). Then

Pressure above *A-A* in the left column $= \rho_2 g h_2 + \rho_1 g h_1 + p$

Pressure head in the right column above *A-A* $= 0$

$$\therefore \quad \rho_2 g h_2 + \rho_1 g h_1 + p = 0$$

$$\therefore \quad p = -(\rho_2 g h_2 + \rho_1 g h_1).$$



(b) For vacuum pressure

Problem 2.9 The right limb of a simple U-tube manometer containing mercury is open to the atmosphere while the left limb is connected to a pipe in which a fluid of sp. gr. 0.9 is flowing. The centre of the pipe is 12 cm below the level of mercury in the right limb. Find the pressure of fluid in the pipe if the difference of mercury level in the two limbs is 20 cm.

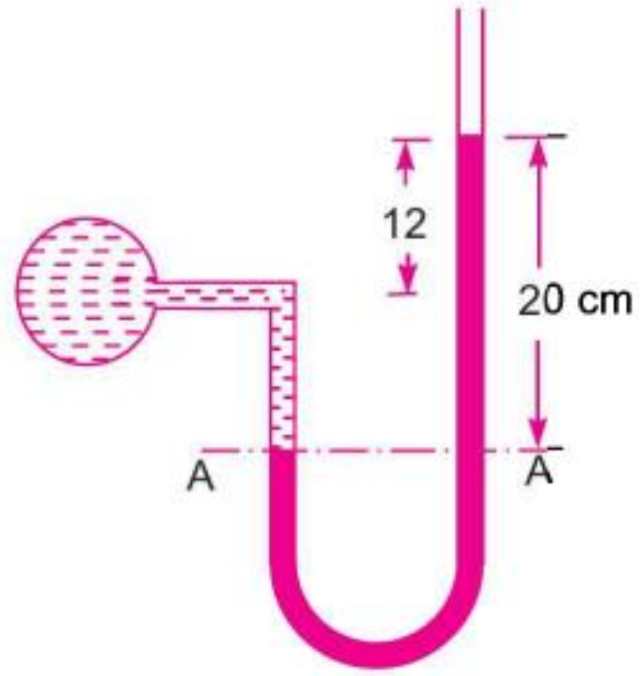


Fig. 2.10

Solution. Given :

Sp. gr. of fluid, $S_1 = 0.9$

$\therefore$ Density of fluid, $\rho_1 = S_1 \times 1000 = 0.9 \times 1000 = 900 \text{ kg/m}^3$

Sp. gr. of mercury, $S_2 = 13.6$

$\therefore$ Density of mercury, $\rho_2 = 13.6 \times 1000 \text{ kg/m}^3$

Difference of mercury level, $h_2 = 20 \text{ cm} = 0.2 \text{ m}$

Height of fluid from A-A, $h_1 = 20 - 12 = 8 \text{ cm} = 0.08 \text{ m}$

Let p = Pressure of fluid in pipe

Equating the pressure above A-A, we get

$$p + \rho_1 g h_1 = \rho_2 g h_2$$

$$p + 900 \times 9.81 \times 0.08 = 13.6 \times 1000 \times 9.81 \times .2$$

$$p = 13.6 \times 1000 \times 9.81 \times .2 - 900 \times 9.81 \times 0.08$$

$$= 26683 - 706 = 25977 \text{ N/m}^2 = \mathbf{2.597 \text{ N/cm}^2}. \textbf{ Ans.}$$

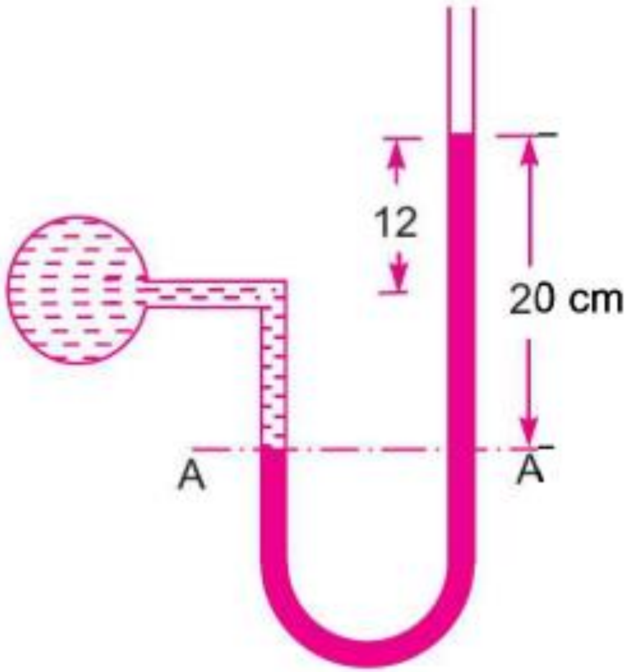
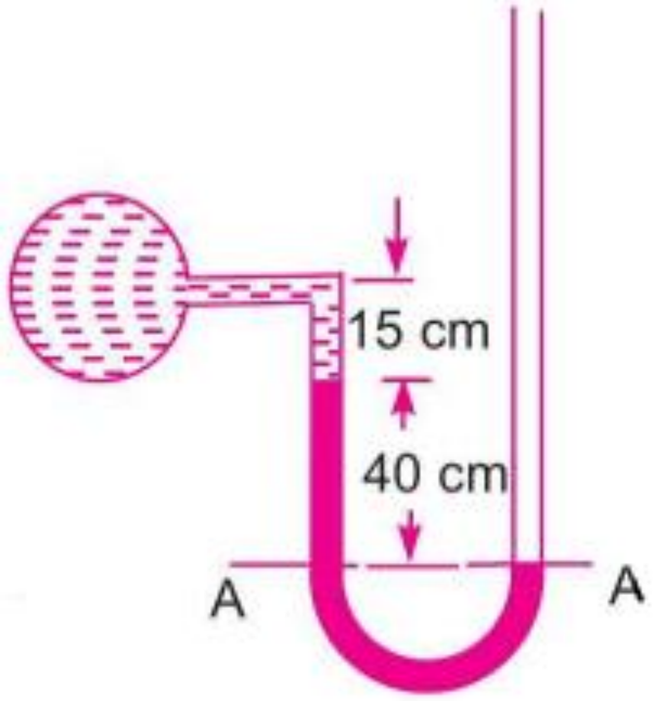


Fig. 2.10

Problem 2.10 A simple U-tube manometer containing mercury is connected to a pipe in which a fluid of sp. gr. 0.8 and having vacuum pressure is flowing. The other end of the manometer is open to atmosphere. Find the vacuum pressure in pipe, if the difference of mercury level in the two limbs is40 cm and the height of fluid in the left from the centre of pipe is 15 cm below.



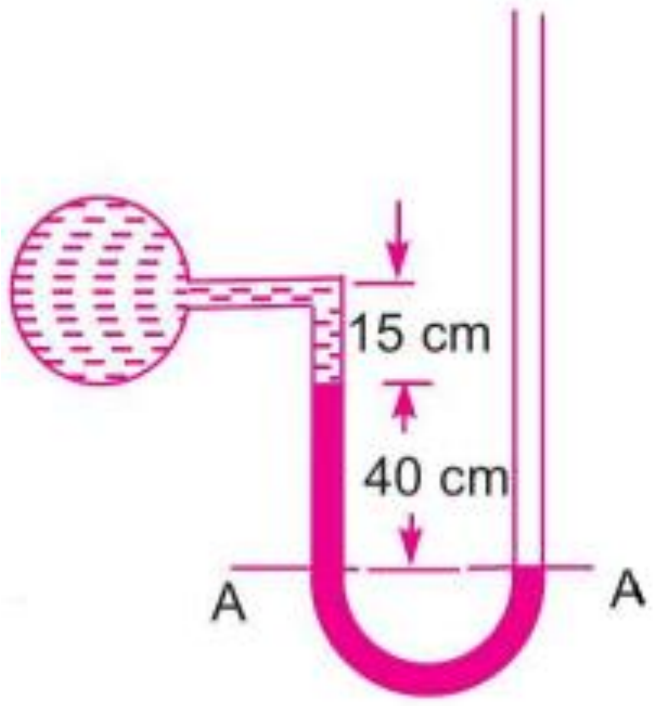
Solution. Given :

Sp. gr. of fluid, $S_1 = 0.8$

Sp. gr. of mercury, $S_2 = 13.6$

Density of fluid, $\rho_1 = 800$

Density of mercury, $\rho_2 = 13.6 \times 1000$



Difference of mercury level, $h_2 = 40$ cm $= 0.4$ m. Height of liquid in left limb, $h_1 = 15$ cm $= 0.15$ m. Let the pressure in pipe $= p$. Equating pressure above datum line A-A, we get

$$\rho_2 g h_2 + \rho_1 g h_1 + p = 0$$

$$p = -[\rho_2 g h_2 + \rho_1 g h_1]$$

$$= -[13.6 \times 1000 \times 9.81 \times 0.4 + 800 \times 9.81 \times 0.15]$$

$$= -[53366.4 + 1177.2] = -54543.6 \text{ N/m}^2 = \mathbf{-5.454 \text{ N/cm}^2}.$$ **Ans.**

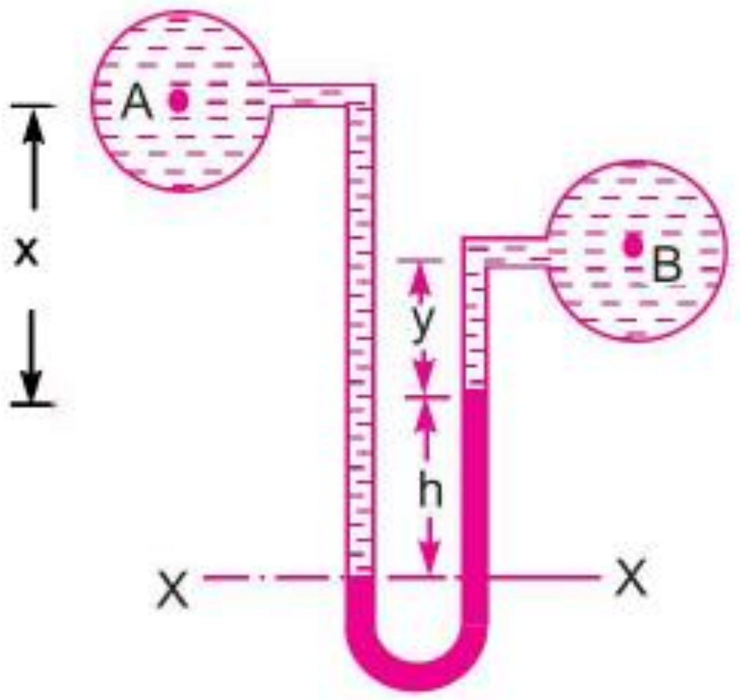
DIFFERENTIAL MANOMETERS : Differential manometers are the devices used for measuring the difference of pressures between two points in a pipe or in two different pipes.

A differential manometer consists of a U-tube, containing a heavy liquid, whose two ends are connected to the points, whose difference of pressure is to be

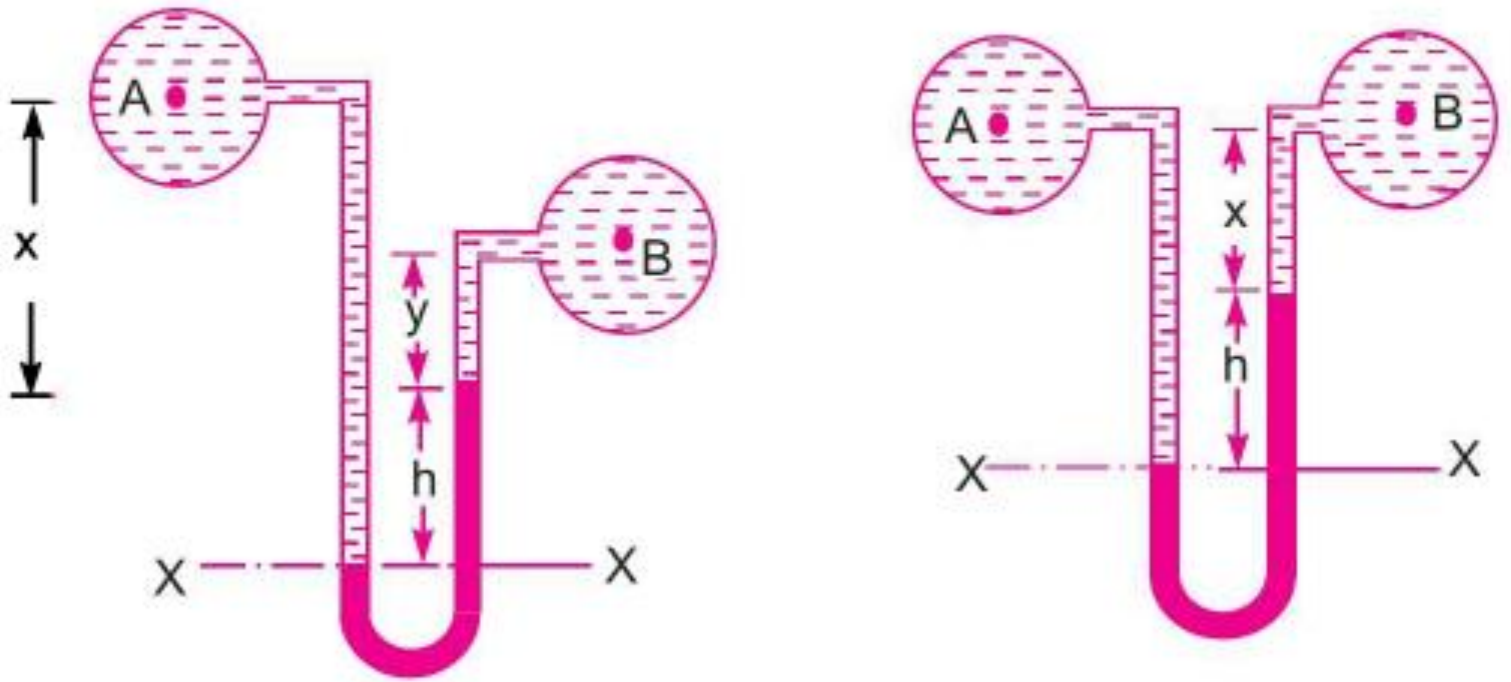
measured. Most commonly types of differential manometers are :

1. U-tube differential manometer and

2. Inverted U-tube differential manometer.



(a)Two pipes at different levels



(a)Two pipes at different levels

(b) A and B are at the same level

Fig. 2.18 *U-tube differential manometers.*

Inverted U-tube Differential Manometer. It consists of an inverted U-tube, containing a light liquid. The two ends of the tube are connected to the points whose difference of pressure is to be measured. It is used for measuring difference of low pressures. Figure shows an inverted U-tube differential manometer connected to the two points A and B. Let the pressure at A is more than the pressure at B.

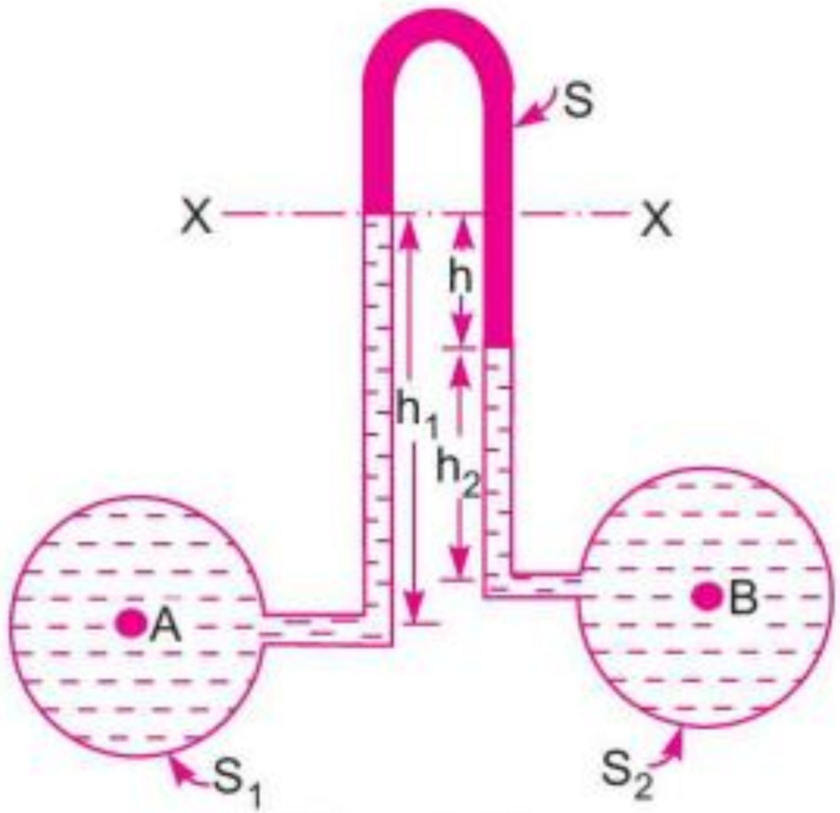


Fig. 2.21

Hydrostatic forces on the surfaces

Hydrostatic force (or) Total pressure force

This concept is used for the design of hydraulic structures like dams ,Hydraulic gates, Ships etc

This concept deals with the fluids (i.e., liquids and gases) at rest. This means that there will be no relative motion between adjacent or neighbouring fluid layers.

$$\text{The velocity gradient} = \frac{\text{change of velocity between two adjacent fluid layers}}{\text{distance between the layers}}$$

$$= \frac{du}{dy} = 0 \text{ will be zero}$$

Hydrostatic force on the depth of the container and surface area for the Given fluid

The shear stress which is equal to Zero on the fluid particles will be :

1. Due to pressure of fluid normal to the surface,
2. Due to gravity (or self-weight of fluid particles).

TOTAL PRESSURE AND CENTRE OF PRESSURE:

Total pressure is defined as the force exerted by a static fluid on a surface either plane or curved when the fluid comes in contact with the surfaces. This force always acts normal to the surface. Centre of pressure is defined as the point of application of the total pressure on the surface.

There are four cases of submerged surfaces on which the total pressure force and centre of pressure is to be determined.

The submerged surfaces may be :

1. Vertical plane surface,
2. Horizontal plane surface,
3. Inclined plane surface, and
4. Curved surface.

VERTICAL PLANE SURFACE SUBMERGED IN LIQUID

Consider a plane vertical surface of arbitrary shape immersed in a liquid as shown in Figure

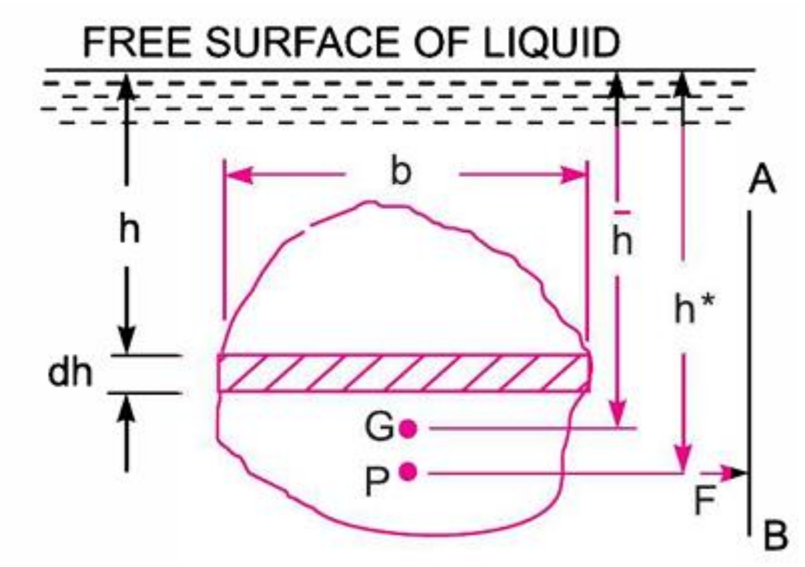
Let A = Total area of the surface

$\bar{h}$ = Distance of C.G. of the area from free surface of liquid

G = Centre of gravity of plane surface

P = Centre of pressure

h* = Distance of centre of pressure from free surface of liquid.



(a) Total Pressure (F): The total pressure on the surface may be determined by dividing the entire surface into a number of small parallel strips. The force on small strip is then calculated and the total pressure force on the whole area is calculated by integrating the force on small strip.

Consider a strip of thickness **dh** and width **b** at a depth of h from free surface of liquid as shown in Figure

Pressure intensity on the strip, p = ρgh

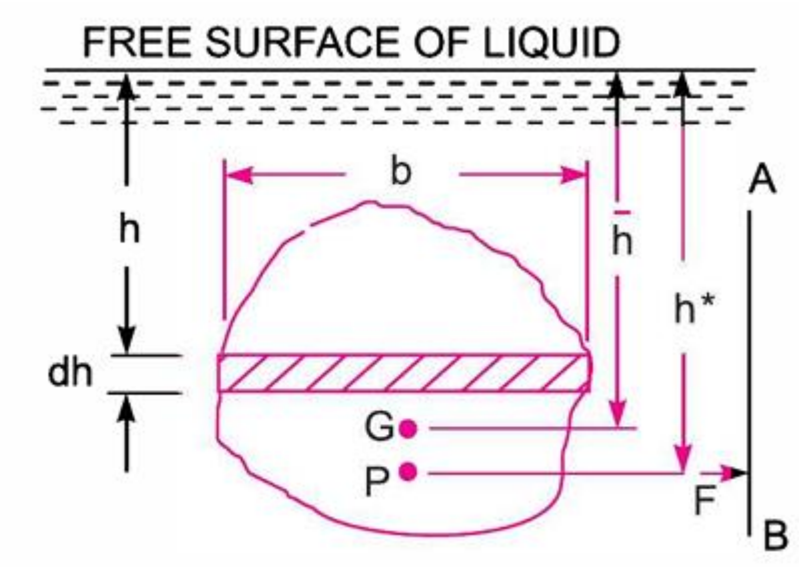
Area of the strip, dA=b x dh

Total pressure force on strip, dF = p X Area

=ρgh x b x dh

Total pressure force on the whole surface,

$$F = \int dF = \int \rho g h \times b \times dh = \rho g \int b \times h \times dh$$



$$\int b \times h \times dh = \int h \times dA$$

= Moment of surface area about the free surface of liquid

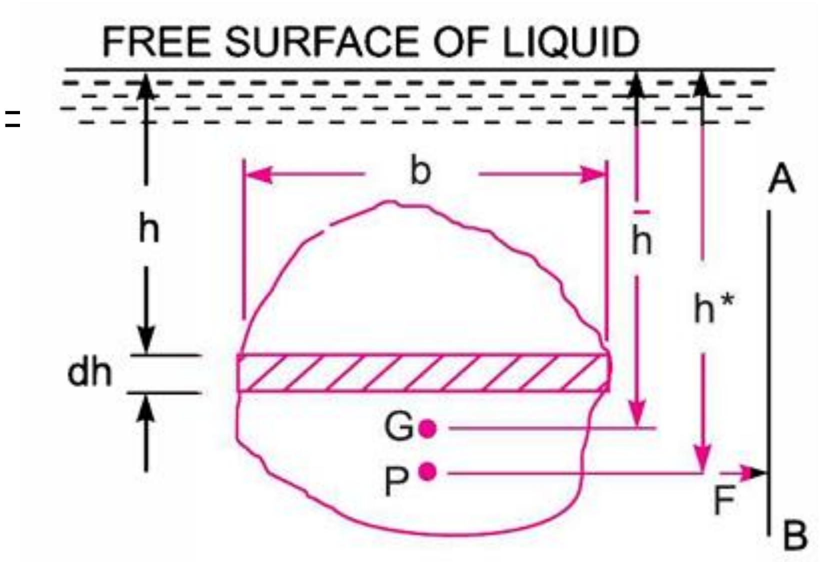
= Area of surface × Distance of C.G. from free surface

$$= A \times \bar{h}$$

$$F = \rho g A \bar{h}$$

(b) Centre of Pressure (h*) : Centre of pressure is calculated by using the "Principle of Moments", which states that the moment of the resultant force about an axis is equal to the sum of moments of the components about the same axis.

The resultant force F is acting at P, at a distance h* from free surface of the liquid as shown in Figure.



Hence moment of the force F about free surface of the liquid =

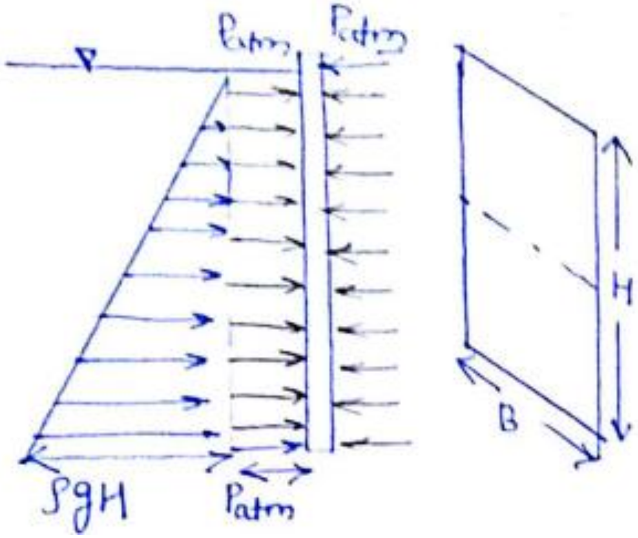
Moment of force dF, acting on a strip about free surface of liquid = dF x h

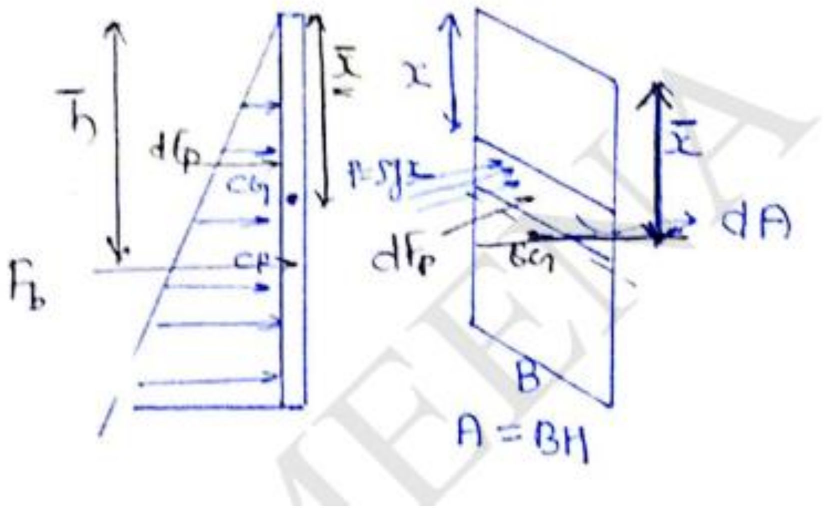
$$\{ dF = pgh \times b \times dh \}$$

$$= pgh \times b \times dh \times h$$

Hydrostatic Force :(Total Pressure Force)

The concept is used to design for Hydraulic Structures ,Hydraulic gates, dams , Ships and submarines….etc.





$$F_p = P \times A$$

$$dF_p = P\,dA$$

$$\int dF_p = \rho g \int x\,dA$$

$$F_p = \rho g \int x\,dA$$

$$F_p = \omega A \bar{x}$$

$$A\bar{x} = \int x\,dA$$

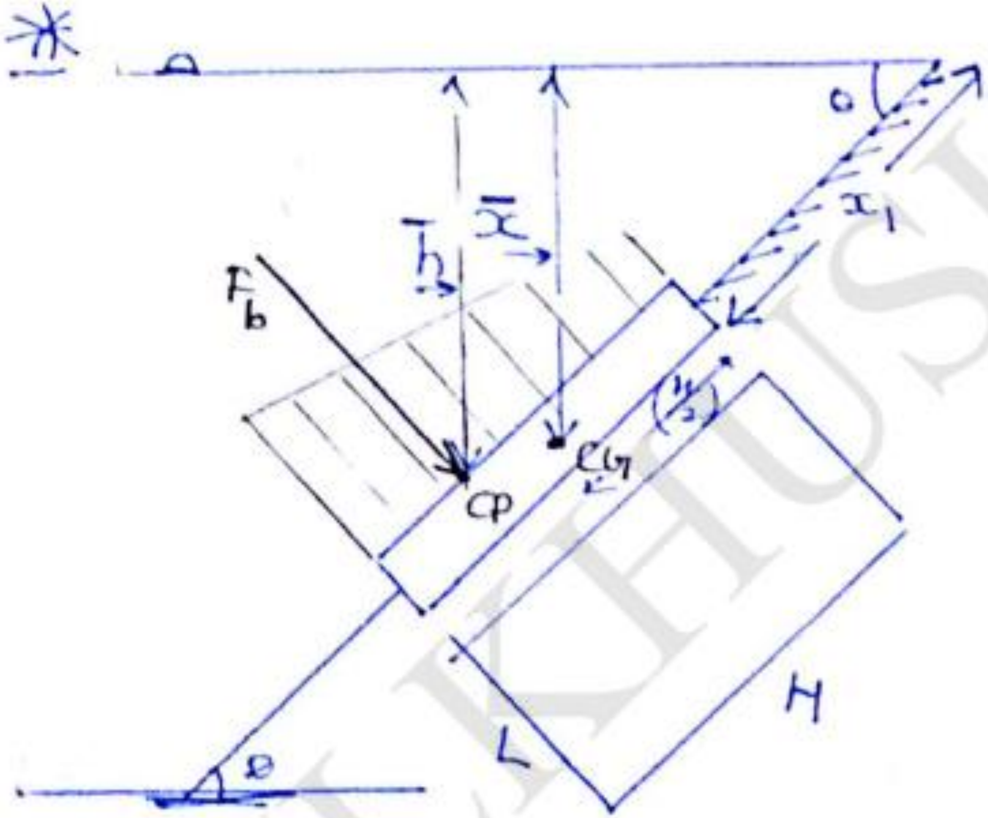
Hydrostatic force = Net force exerted by the fluid on the body is known as total pressure force for flat surface in any Orientation

$$F_r = wA\bar{x}$$

Centre of pressure (CP) : The point of application of total pressure force is known as Centre of pressure.it is measured in vertical depth from the free surface

$$\bar{h} = \frac{I_{GG}\sin^2\theta}{A\bar{x}} + \bar{x}$$

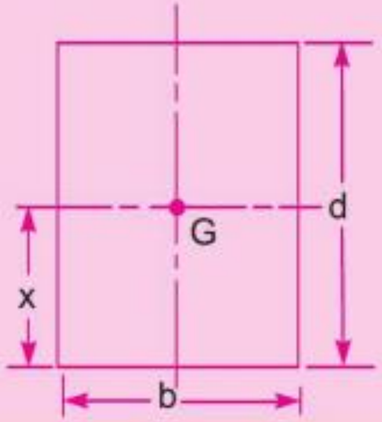
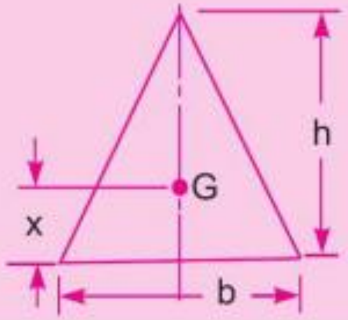
Igg = MOI of body about axis parallel to the to pressure θ is the angle of flat surface from the free surface / Horizontal surface.

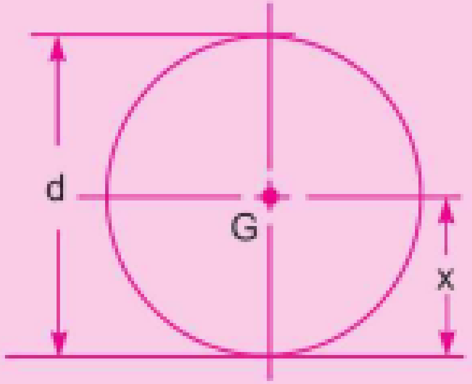
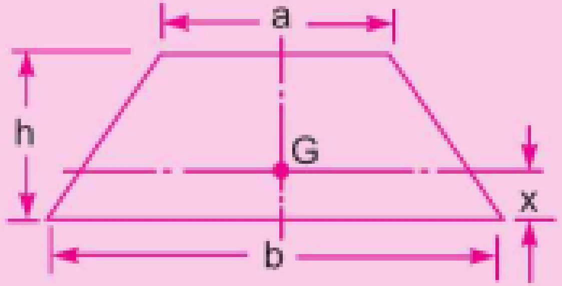


$$F_p = wA\bar{x}$$

$$\bar{h} = \frac{I_{GG}\sin^2\theta}{A\bar{x}} + \bar{x}$$

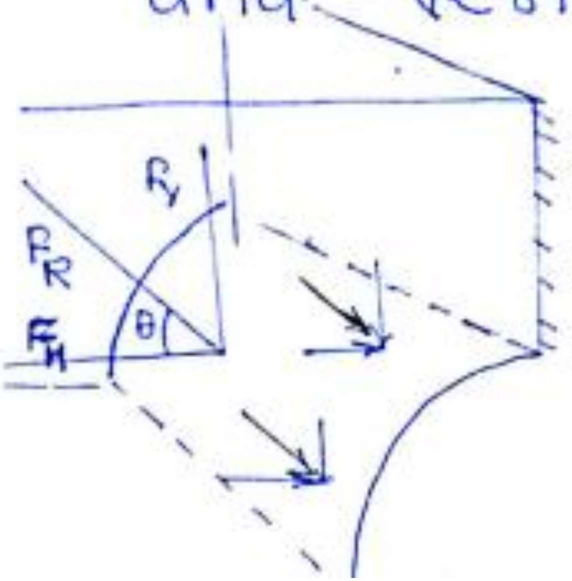
$$I_{GG} = \frac{LH^3}{12}$$

Plane surface	*C.G. from the base*	*Area*	*Moment of inertia about an axis passing through C.G. and parallel to base* (I_G)	*Moment of inertia about base* (I_0)
1. Rectangle 	$x = \frac{d}{2}$	bd	$\frac{bd^3}{12}$	$\frac{bd^3}{3}$
2. Triangle 	$x = \frac{h}{3}$	$\frac{bh}{2}$	$\frac{bh^3}{36}$	$\frac{bh^3}{12}$

Plane surface	*C.G. from the base*	*Area*	*Moment of inertia about an axis passing through C.G. and parallel to base* (I_G)	*Moment of inertia about base* (I_0)
3. Circle 	$x = \frac{d}{2}$	$\frac{\pi d^2}{4}$	$\frac{\pi d^4}{64}$	—
4. Trapezium 	$x = \left(\frac{2a+b}{a+b}\right)\frac{h}{3}$	$\frac{(a+b)}{2} \times h$	$\left(\frac{a^2+4ab+b^2}{36(a+b)}\right) \times h^3$	—

Hydrostatic forces on Curved surface:—

Hydrostatic forces of curved surface are in different orientation so the resultant force can be obtain by horizontal and vertide component.



$$F_R = \sqrt{F_H^2 + F_V^2}$$

$$\tan\theta = \frac{F_V}{F_H}$$

Buoyancy and Floatation

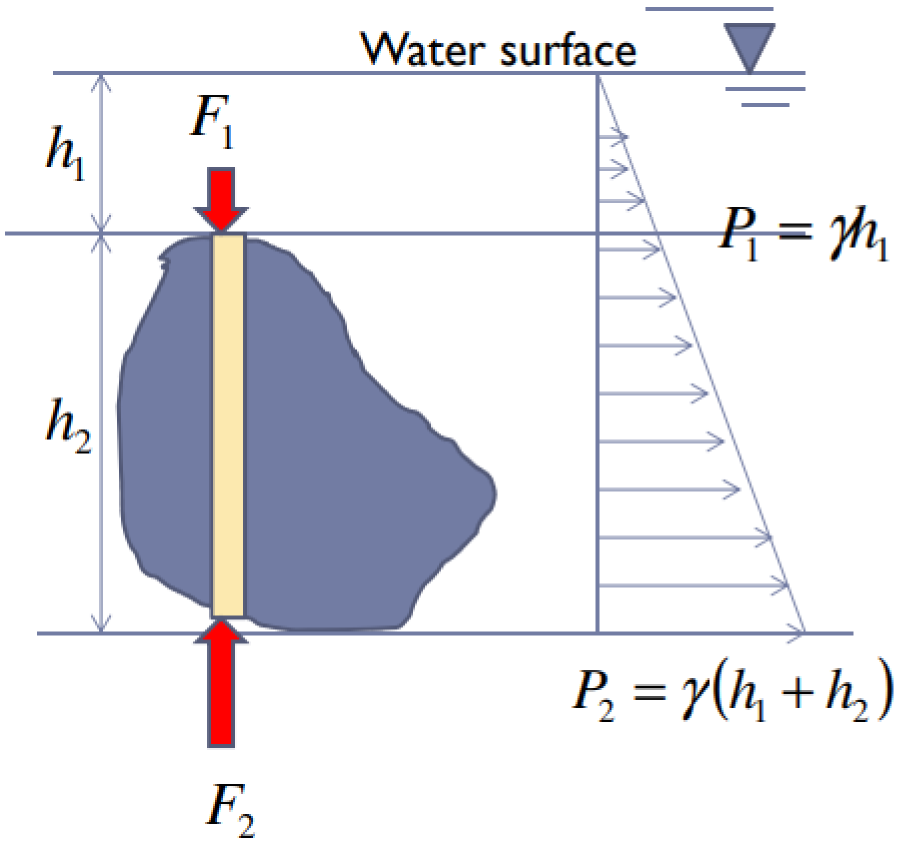
- When a body is immersed wholly or partially in a fluid, it is subjected to an upward force which tends to lift (buoy)it up.
- The tendency of immersed body to be lifted up in the fluid due to an upward force opposite to action of gravity is known as **buoyancy.**
- The force tending to lift up the body under such conditions is known as **buoyant force** or force of buoyancy or up-thrust.

The magnitude of the buoyant force can be determined by Archimedes' principle which states

When a body is immersed in a fluid either wholly or partially, it is buoyed or lifted up by a force which is equal to the weight of fluid displaced by the body"

Buoyancy and Floatation

Lets consider a body submerged in water as shown in figure.



The force of buoyancy "resultant upward force or thrust exerted by fluid on submerged body" is given

$$F_B = F_2 - F_1$$

$$F_B = \gamma(h_1 + h_2)dA - \gamma(h_1)dA$$

$$F_B = \gamma[(h_2)dA]$$

$$F_B = \gamma[volume]$$

dA=Area of cross-section of element

γ= Specific weight of liquid

- $F_B = \gamma[volume]$ = Weight of volume of liquid displaced by the body (Archimedes's Principle)

- Force of buoyancy can also be determined as difference of weight of a body in air and in liquid.

Let

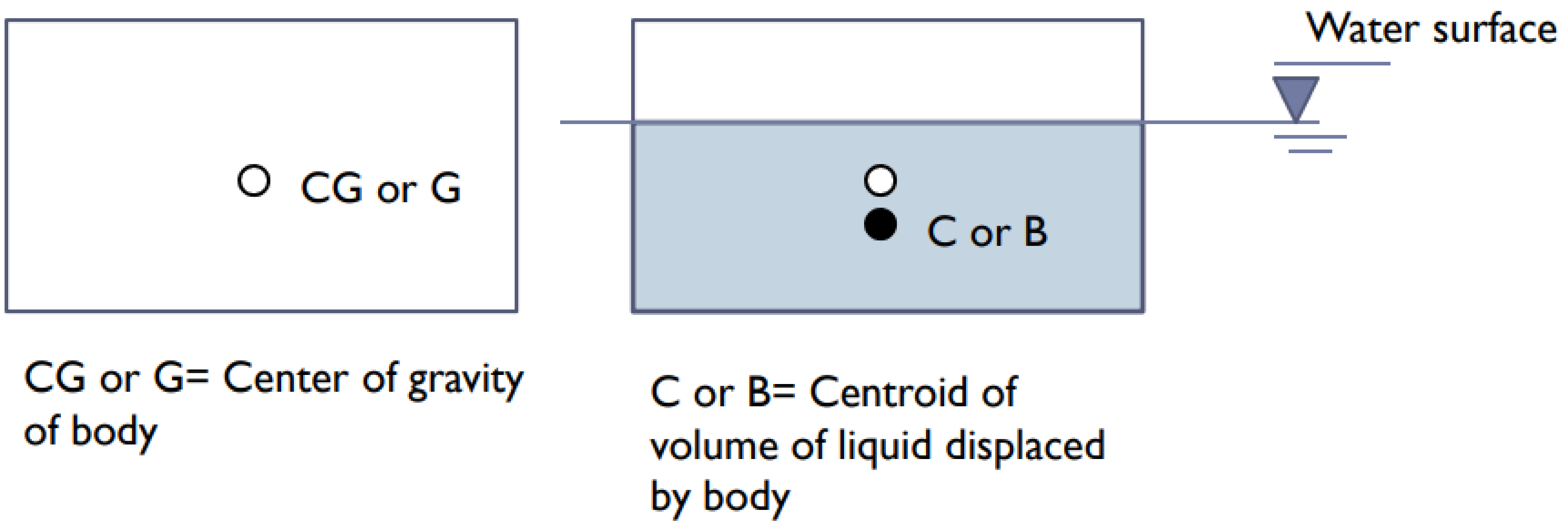
$W_a =$ weight of body in air

$W_l =$ weight of body in liquid

$F_B = W_a - W_l$

Center of Buoyancy (B): The point of application of the force of buoyancy on the body is known as the center of buoyancy.

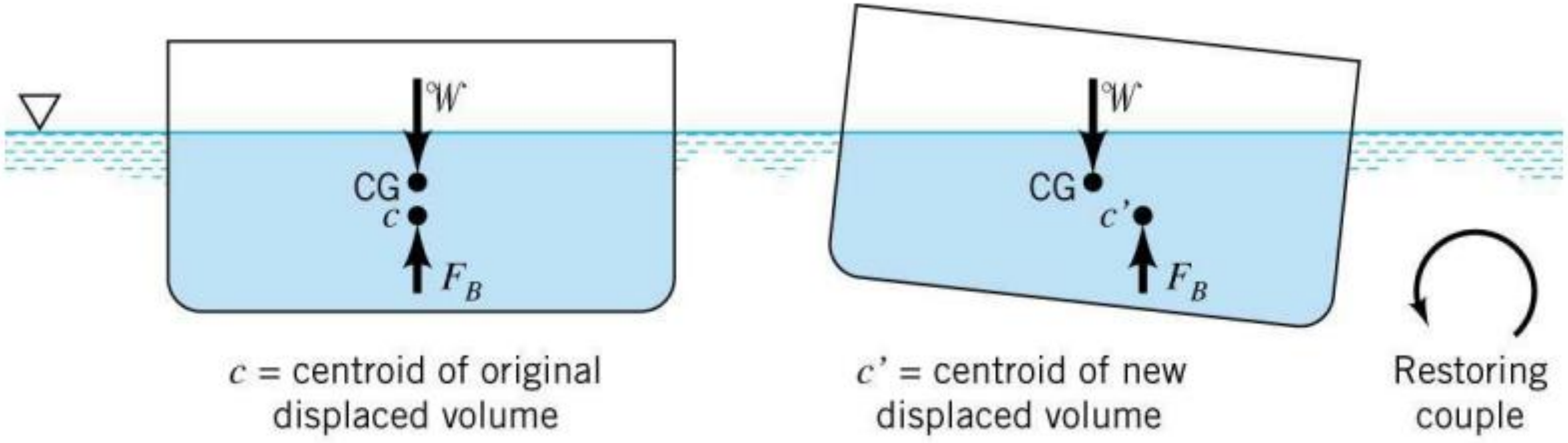
It is always the center of gravity of the volume of fluid displaced.



Types of equilibrium of Floating Bodies

Stable Equilibrium:

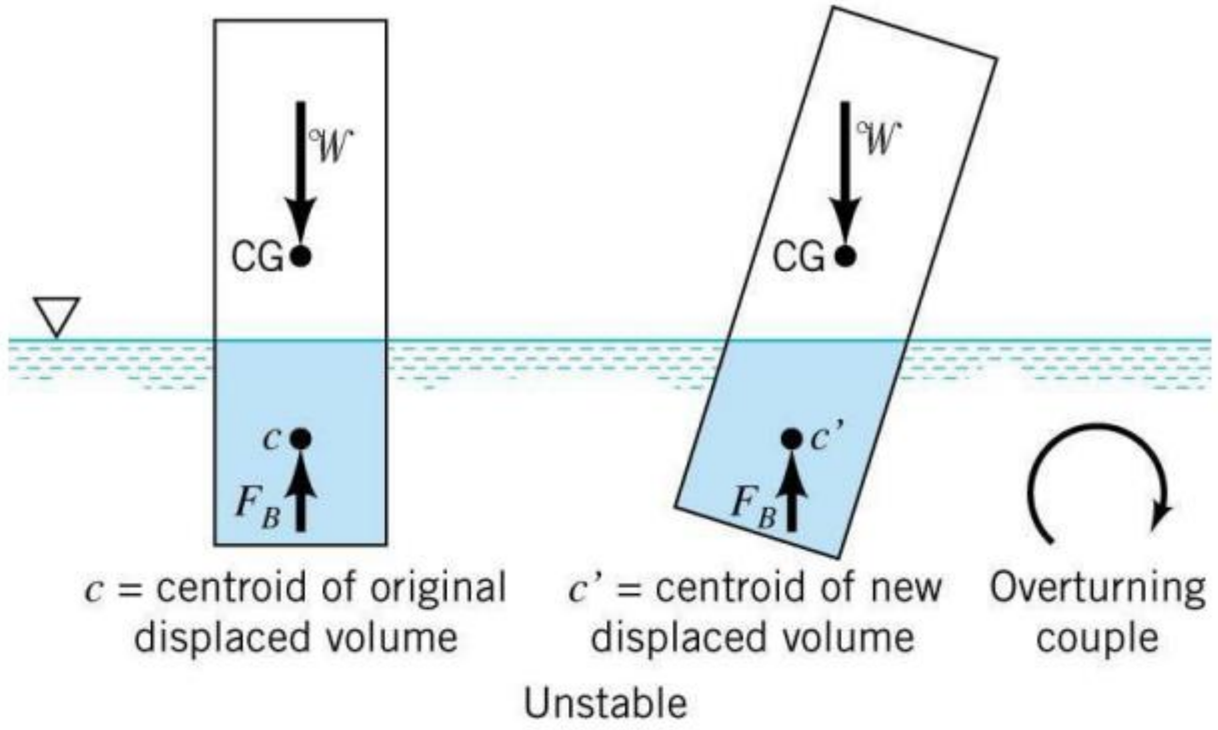
If a body returns back to its original position due to internal forces from small angular displacement, by some external force, then it is said to be in stable equilibrium.



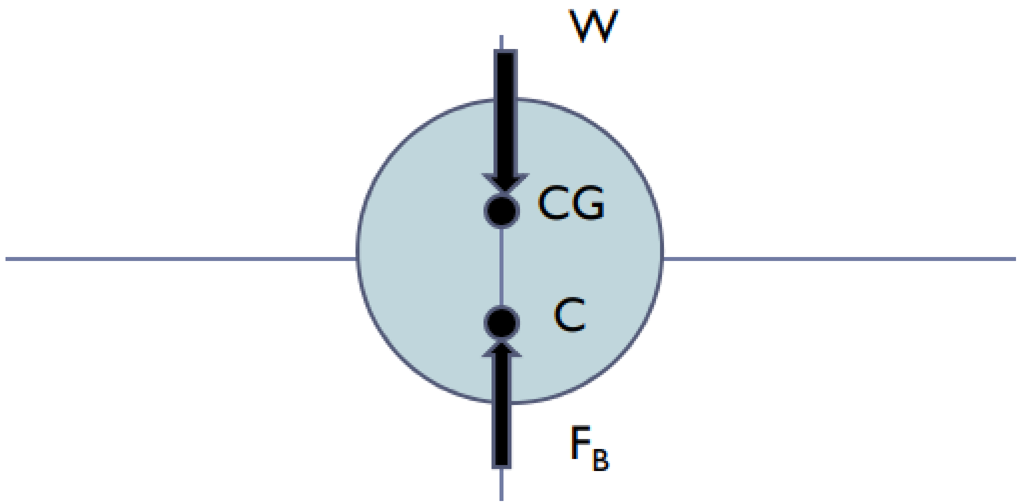
Stable

Note: Center of gravity of the volume (centroid) of fluid displaced is also the center of buoyancy

Unstable Equilibrium: If the body does not return back to its original position from the slightly displaced angular displacement and heels farther away, then it is said to be in unstable equilibrium



Neutral Equilibrium: If a body, when given a small angular displacement, occupies new position and remains at rest in this new position, it is said to be in neutral equilibrium.

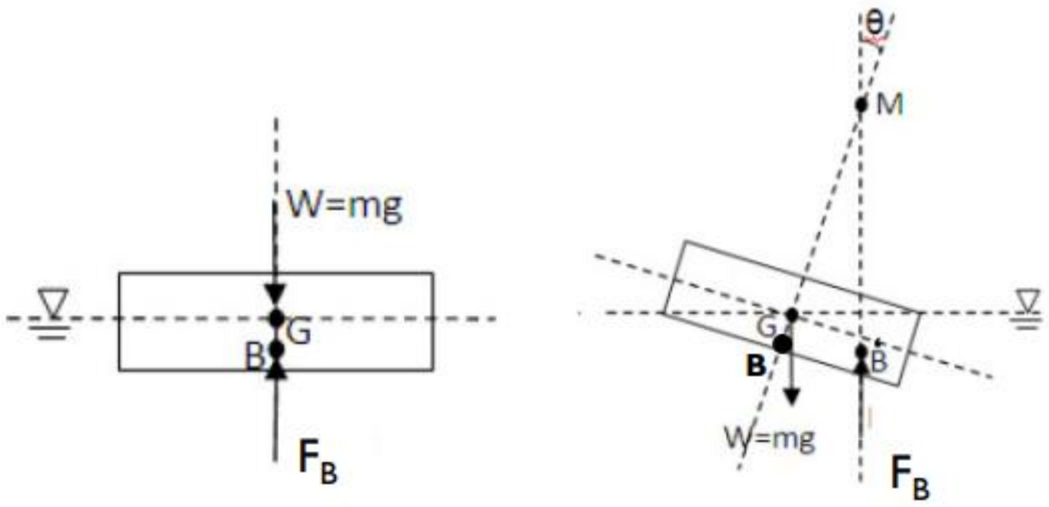


Metacentre and Metacentric Height

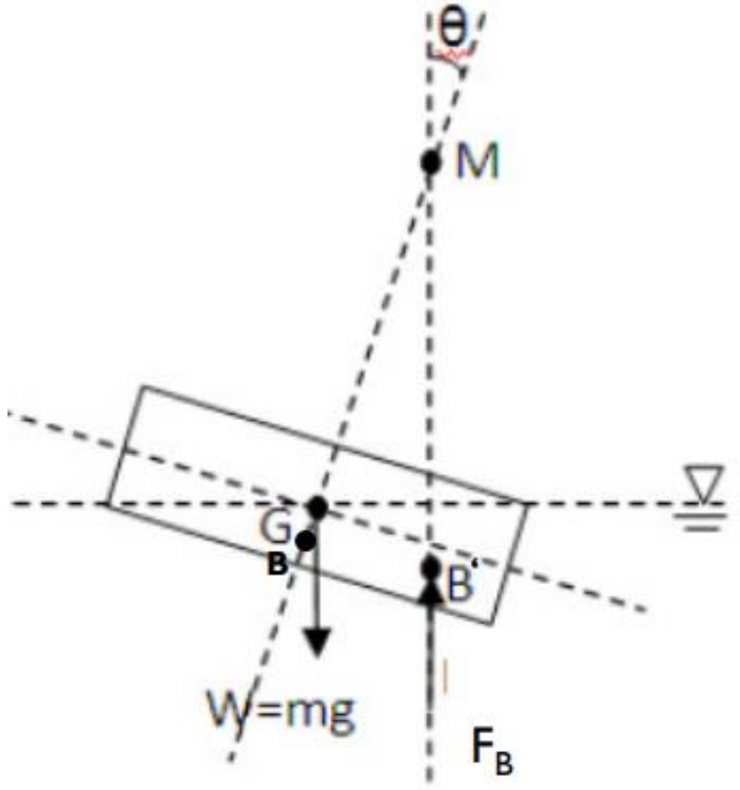
Center of Buoyancy (B) The point of application of the force of buoyancy on the body is known as the center of buoyancy

Metacenter (M): The point about which a body in stable equilibrium start to oscillate when given a small angular displacement is called metacenter.

It may also be defined as point of intersection of the axis of body passing through center of gravity (CG or G) and original center of buoyancy (B) and a vertical line passing through the center of buoyancy (B') of tilted position of body.



Metacentric height (GM): The distance between the center of gravity (G) of floating body and the metacenter (M) is called metacentric height. (i.e., distance GM shown in fig)



GM=BM-BG

Condition of Stability

For Stable Equilibrium

Position of metacenter (M) is above than center of gravity (G)

For Unstable Equilibrium

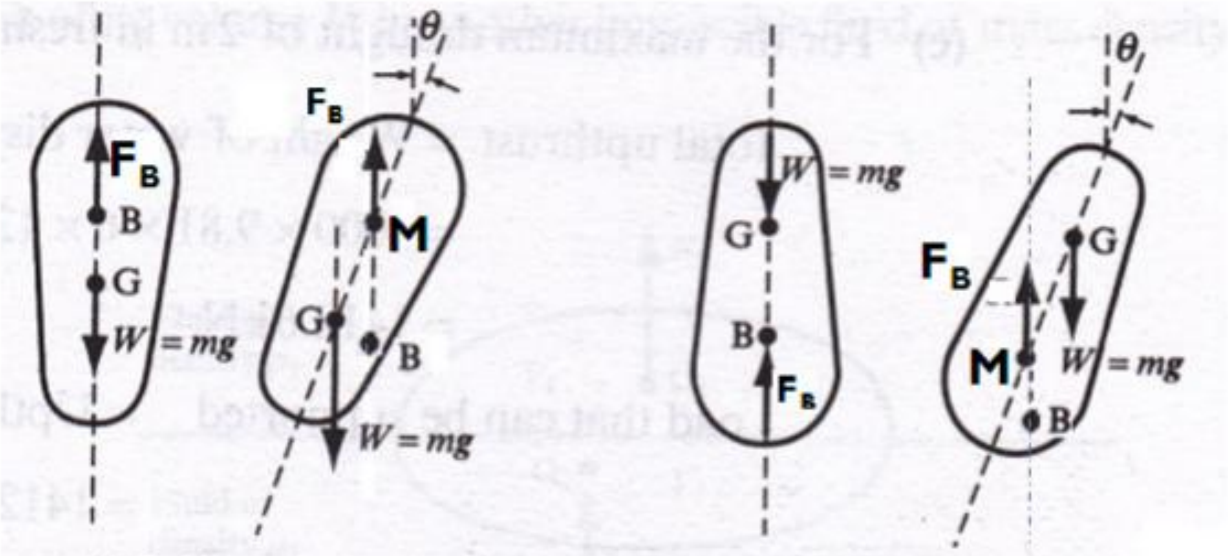
Position of metacenter (M) is below than center of gravity (G)

For Neutral Equilibrium

Position of metacenter (M) coincides center of gravity (G)

Overturning moment

Restoring moment

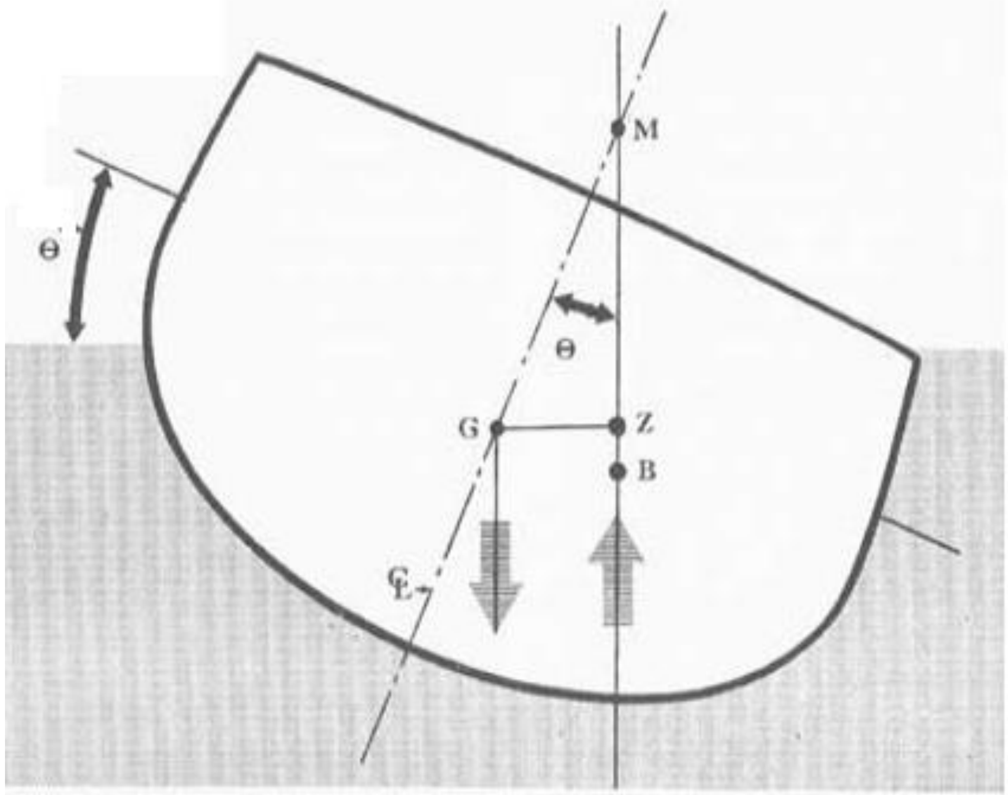


(a) Stable (b) Unstable

Determination of Metacentric height

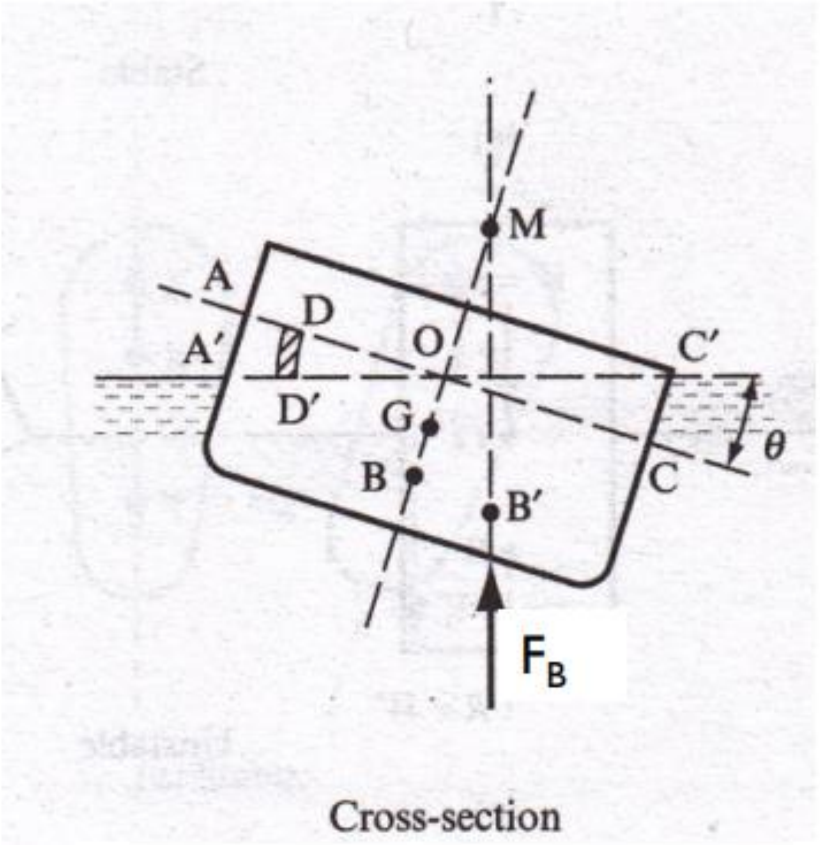
The metacentric height may be determined by the following two methods

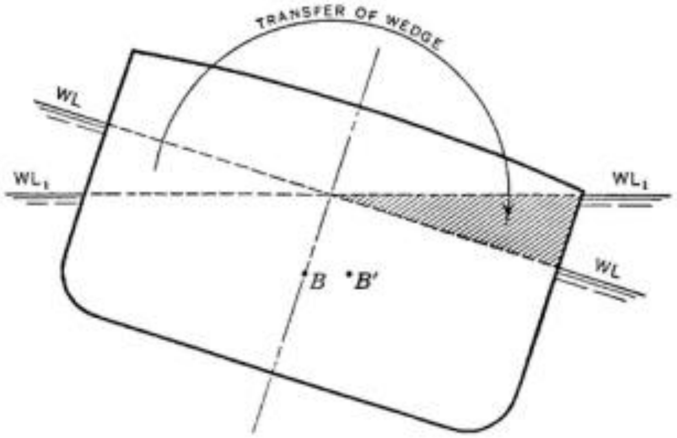
1. Analytical method 2. Experimental method



- In Figure shown AC is the original waterline plane and B the center of buoyancy in the equilibrium position.
- When the vessel is tilted through small angle θ, the center of buoyancy will move to B' as a result of the alteration in the shape of displaced fluid.

A'C' is the waterline plane in the displaced position.





Kinematics of fluid

Kinematics of flow

INTRODUCTION

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics. Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined. In this chapter, the methods of determining velocity and acceleration are discussed.

Motion Characteristics:

- Velocity
- Acceleration
- Pressure
- Density

METHODS OF DESCRIBING FLUID MOTION

The fluid motion is described by two methods.

They are —(i) Lagrangian Method, and

(ii) Eulerian Method.

In the **Lagrangian method**, a single fluid particle is followed during its motion and its velocity, acceleration, density, etc., are described.

In case of **Eulerian method,** the velocity, acceleration, pressure, density etc., are described at a point in flow field. The Eulerian method is commonly used in fluid mechanics.

Fluid motion is described by two methods.

Methods:

Lagrangian Method - Describes a defined mass (position, velocity, acceleration, pressure, temperature etc.,) as functions of time.

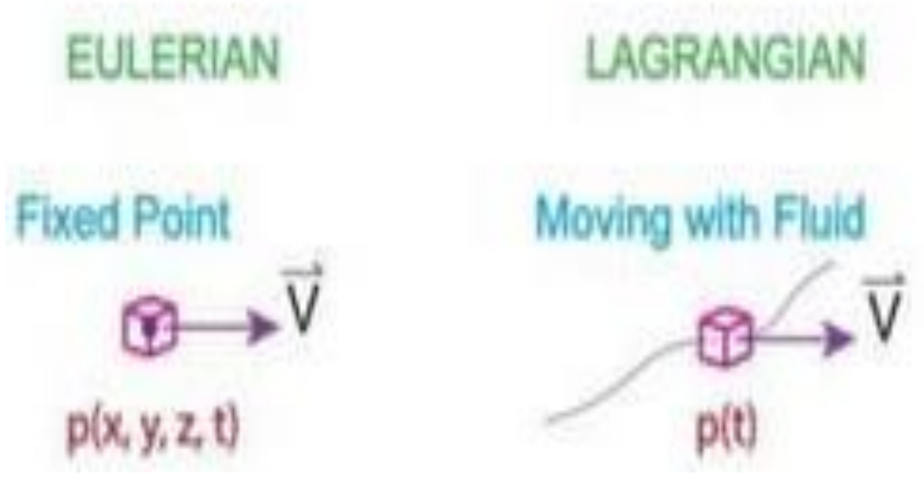
Example:

Track the location of a migrating bird

Eulerian Method - Describes a flow field (velocity, acceleration, pressure, temperature etc.,) as functions of position and time.

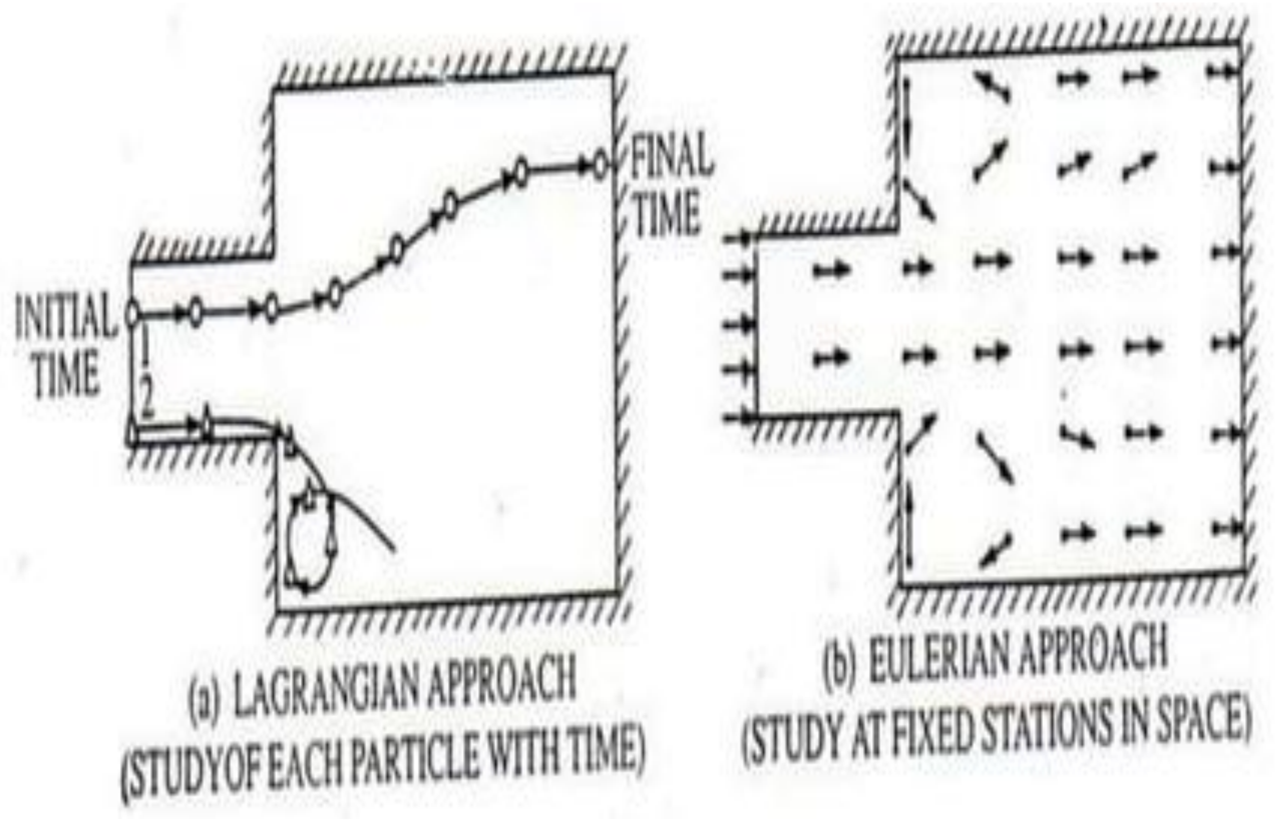
Example:

Count the birds passing through a particular location




Rising at 2 degrees per hour!
Mr. Euler
Dropping at 1 degree per hour!
Mr. Lagrange

- In the Eulerian approach, the fluid motion at all points in the flow field is determined by applying the laws of mechanics at all fixed stations.
- This is considerably easier than the Lagrangian approach and is followed in the study of Fluid Mechanics.



Difference Between Lagrangian and Eulerian Description

- Imagine a person standing beside a river measuring its properties.

In the Lagrangian approach, he throws in a probe that moves downstream with the water.

In the Eulerian approach, he anchors the probe at a fixed location in the water.

- **Experimental measurements** are more suitable to the Eulerian Description.

- Equations of motion of fluid flow in Lagrangian description are well defined (Newtons' second law), but needs to be carefully derived for the Eulerian description

Lagrangian method

- In this method, a single fluid particle is chosen and followed during its motion.
- Its velocity, acceleration, density etc. is described with respect to its location in space and time from a fixed position at the start of the motion.
- The position of the fluid particle (x,y,z) at any time t with respect to its position (a,b,c) at time t=0 is given as,

$x=f1(a,b,c,t)$; $y=f2(a,b,c,t)$; $z=f3(a,b,c,t)$

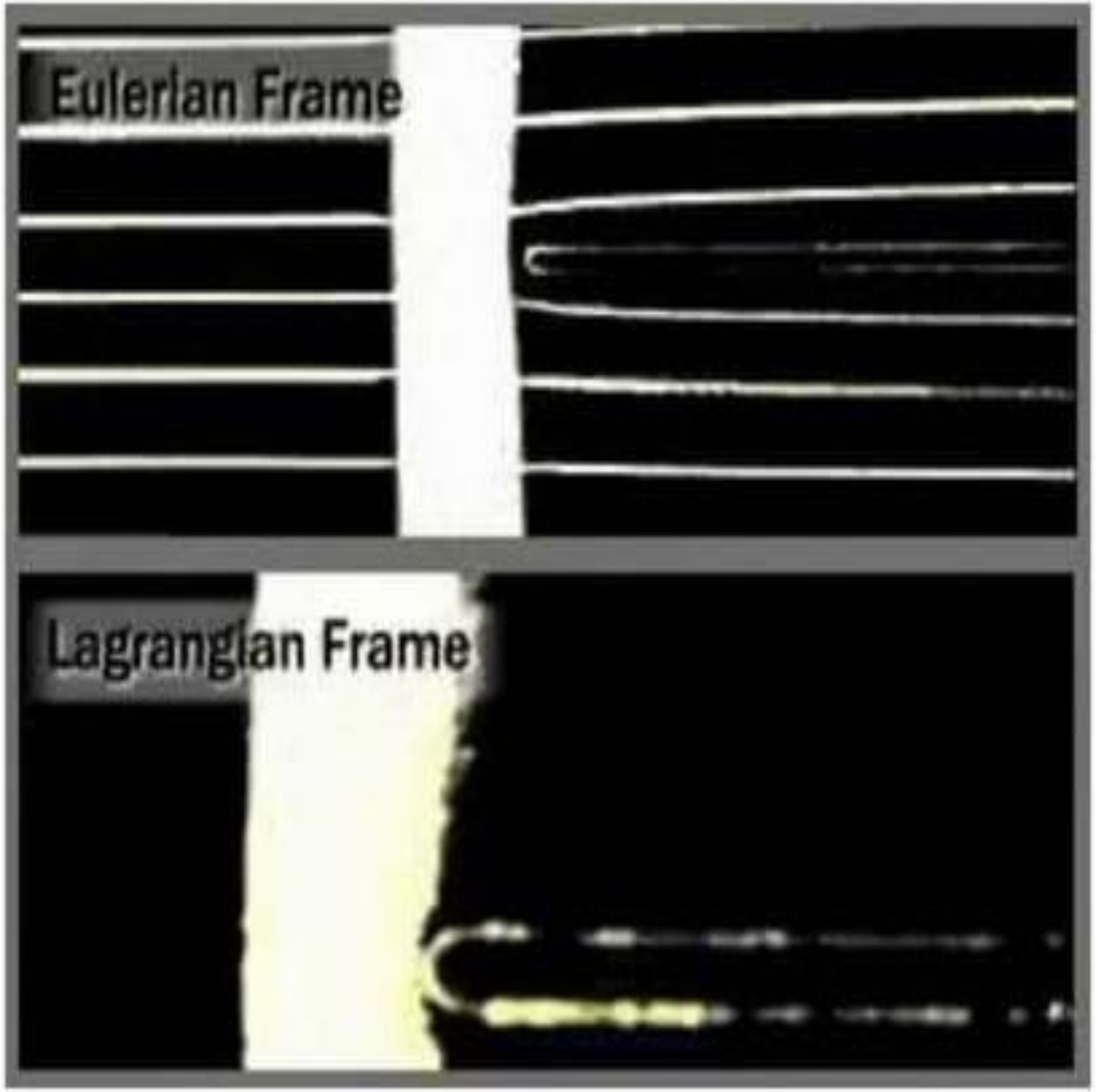
Then, velocity is given by

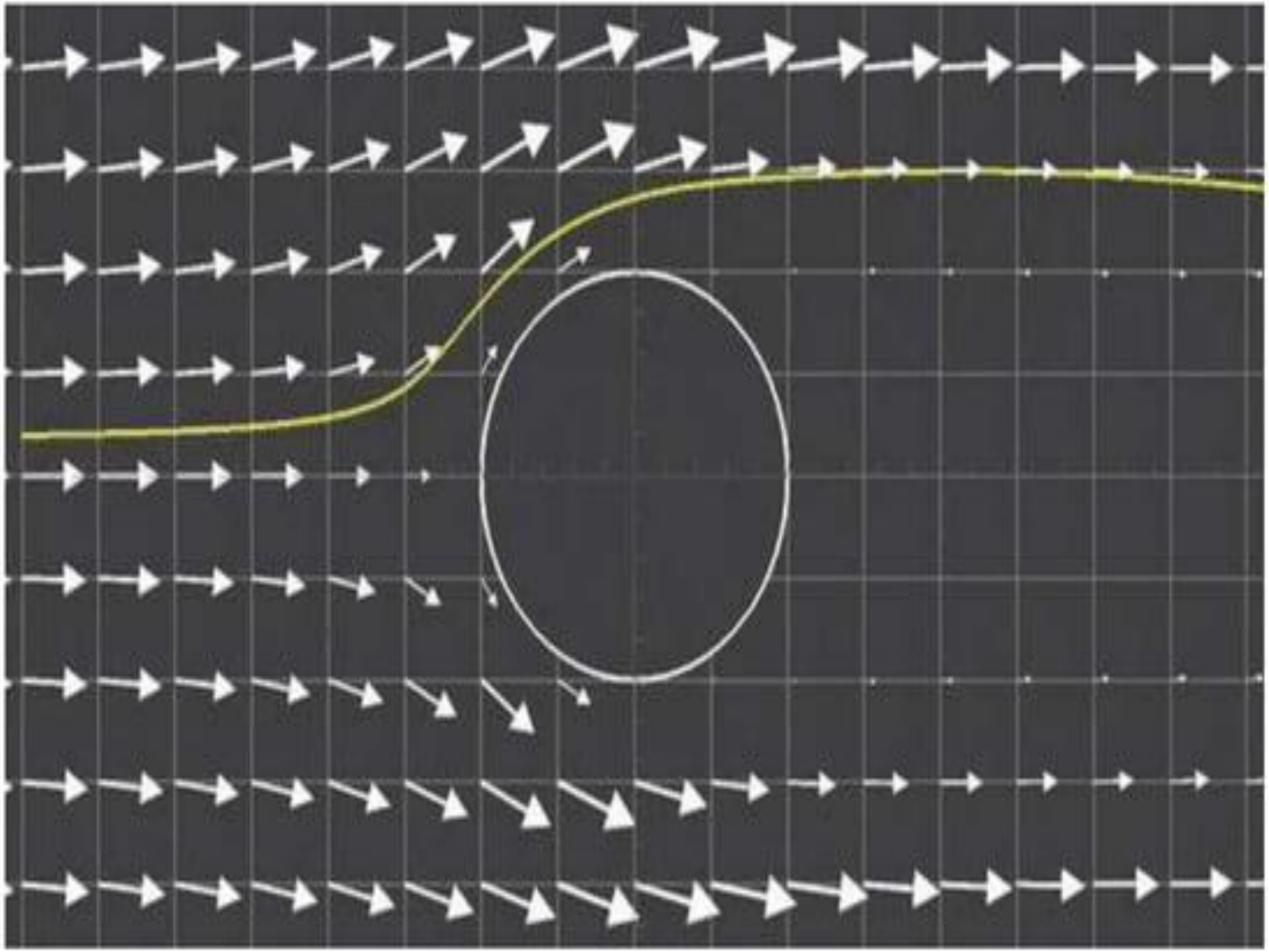
$u=dx/dt$; $v=dy/dt$; $w=dz/dt$

$a_x = du/dt = d^2x/dt^2$; $a_y = d^2y/dt^2$; $a_z = d^2z/dt^2$

Resultant velocity=?

Resultant acceleration=?


Eulerian Frame
Lagrangian Frame



Eulerian method

- In this method, a point or section is chosen in the flow field
- Its velocity, acceleration, density etc. are observed at that point
- It is a commonly used method because of mathematical simplicity Let u, v, and w be the components of resultant velocity in x, y and z axis respectively. The velocity components vary along space and time.

$$u=f1(x,y,z,t);$$

$$v=f2(x,y,z,t);$$

$$w=f3(x,y,z,t)$$

Then, resultant velocity is given by

Resultant velocity=?

Flow Visualization is the Visual Examination of Flow-Field Features.

It is important for both physical experiments and numerical solutions.

Description of Flow Patterns:

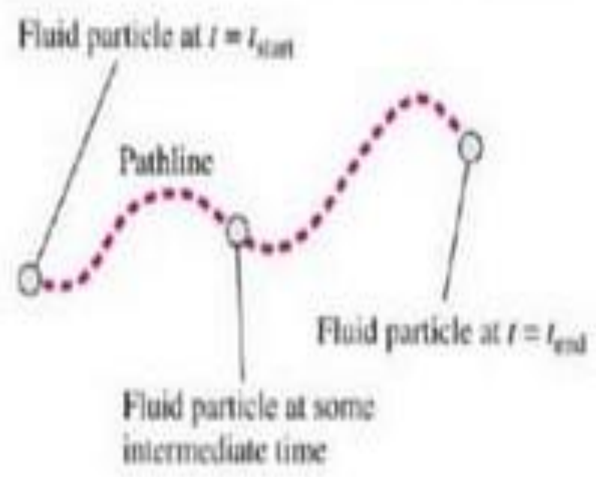
1. Stream Lines

2. Path Lines

3. Streak Lines

4. Stream Tube

- **Path line:** It is trace made by single particle over a period of time.
- **Streamline** show the mean direction of a number of particles at the same instance of time.

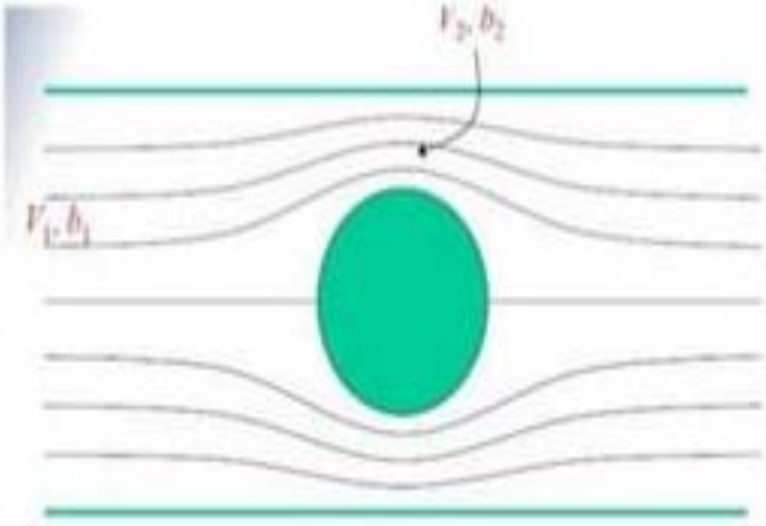


➢ Character of Streamline

1.Streamlines can not cross each other. (otherwise, the cross point will have two tangential lines.)

2. Streamline can't be a folding line, but a smooth curve.

3. Streamline cluster density reflects the magnitude of velocity. (Dense streamlines mean large velocity: while sparse streamlines mean small velocity.)

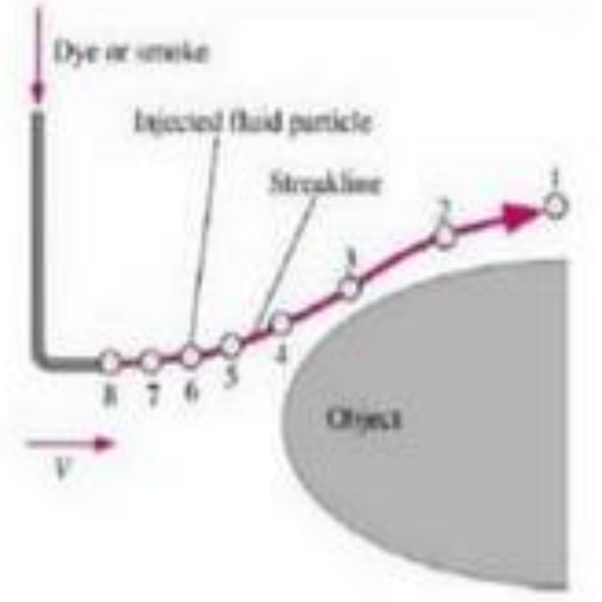


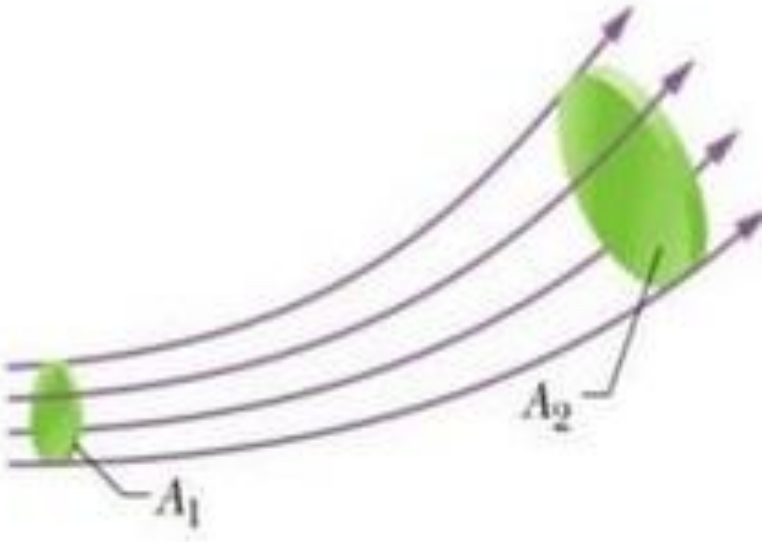
Flow around cylindrical object

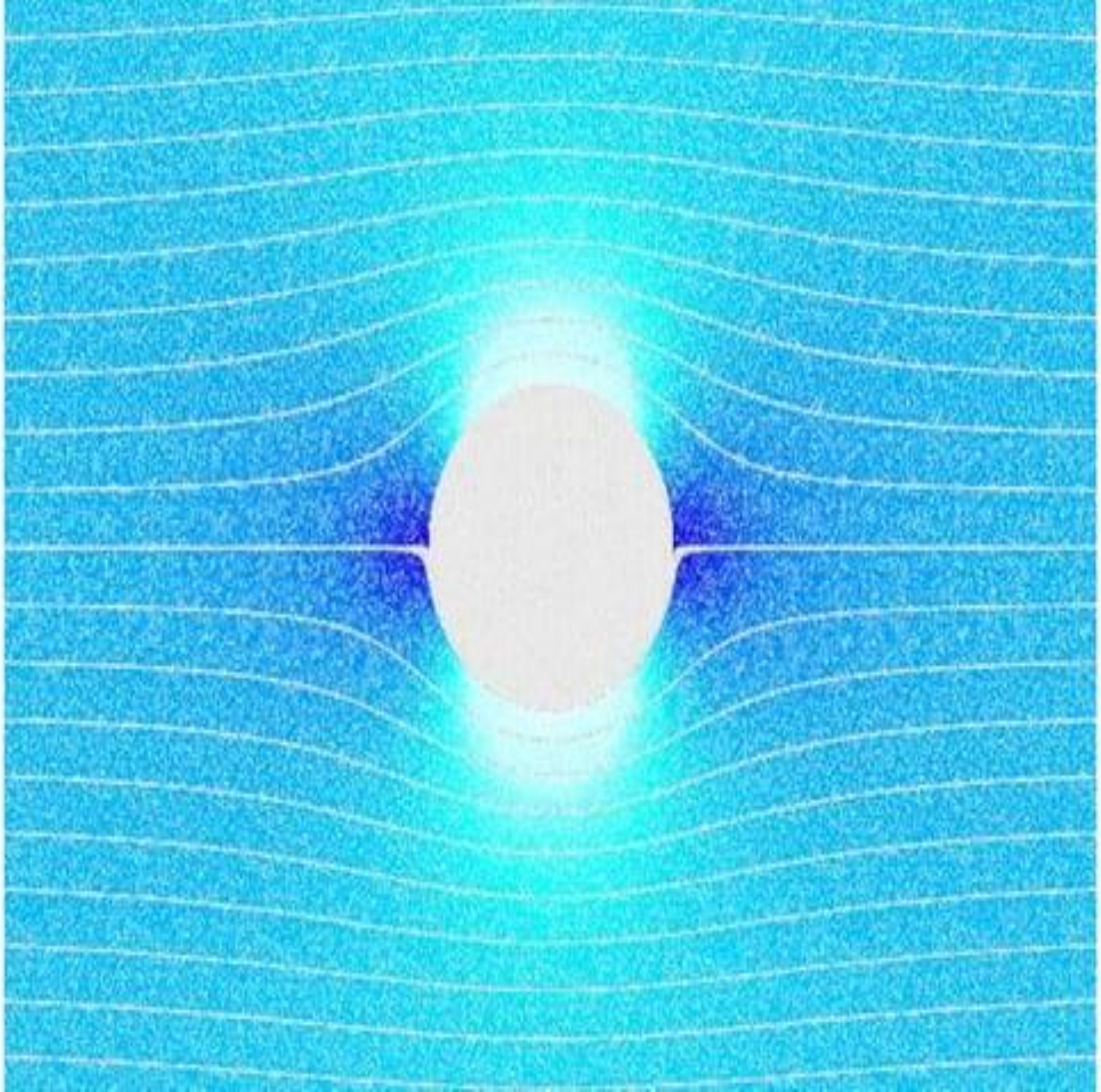
› A **Streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.

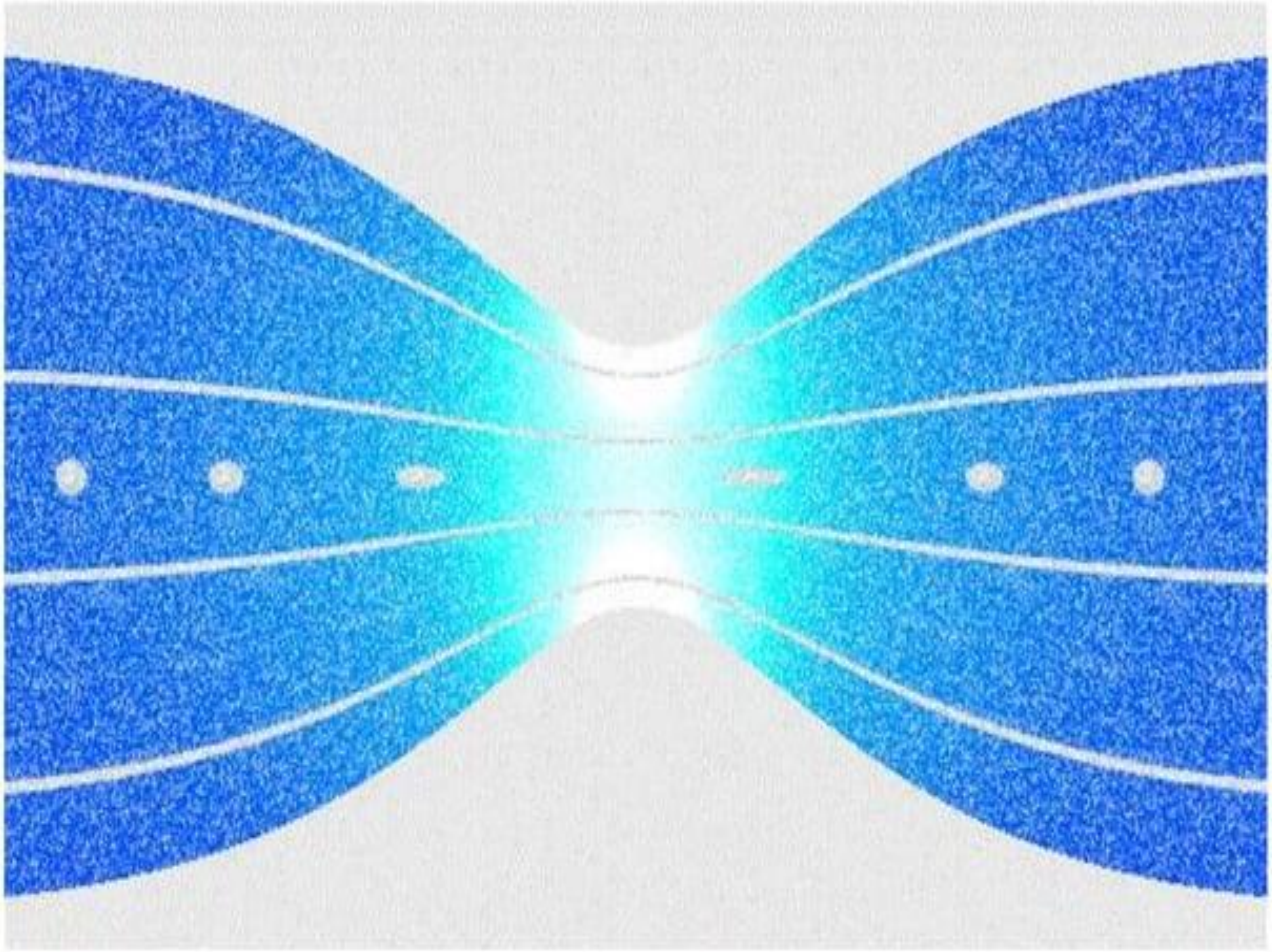
It is an instantaneous picture of the position of all particles in flow that have passed through a given point.

- Streamtube is an imaginary tube whose boundary consists of streamlines.
- The volume flow rate must be the same for all cross sections of the stream tube.









Types of Fluid Flow

1. Steady and Unsteady Flow
2. Uniform and Non-Uniform Flow
3. Laminar and Turbulent Flow
4. Rotational and Irrotational Flow
5. Compressible and Incompressible Flow
6. Ideal and Real Flow
7. One, Two and Three Dimensional Flow

Steady flow is that type of flow in which fluid parameters (velocity, pressure, density etc.) at any point in the flow field do not change with time.

This means that the fluid particles passing through a fixed point have the same flow parameters like velocity, pressure, surface tension etc.

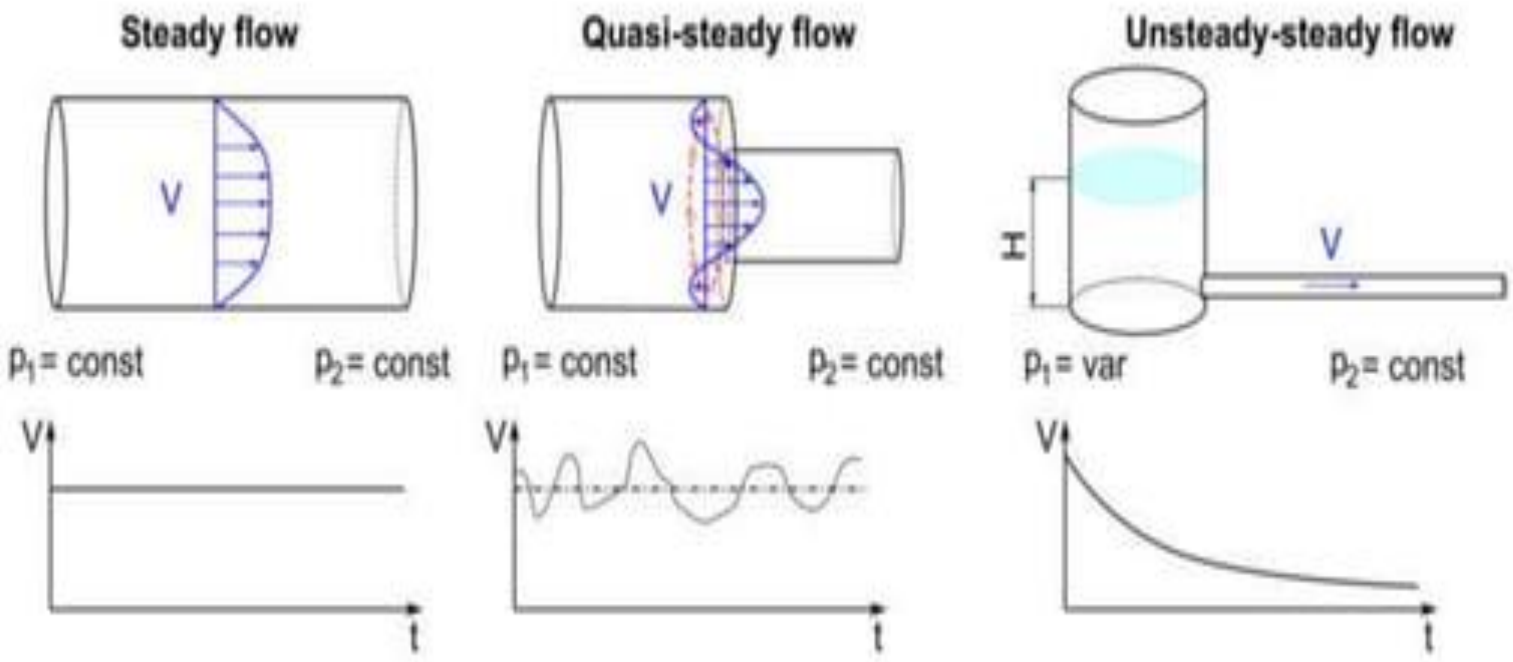
The parameters may be different at the different cross-section of the flow passage.

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

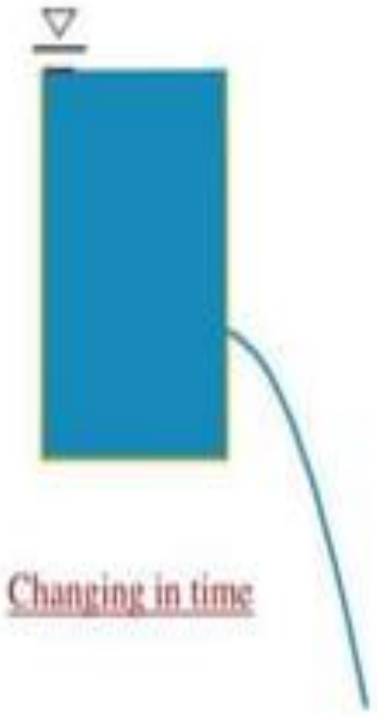
where (x_0, y_0, z_0) is a fixed point in fluid field.

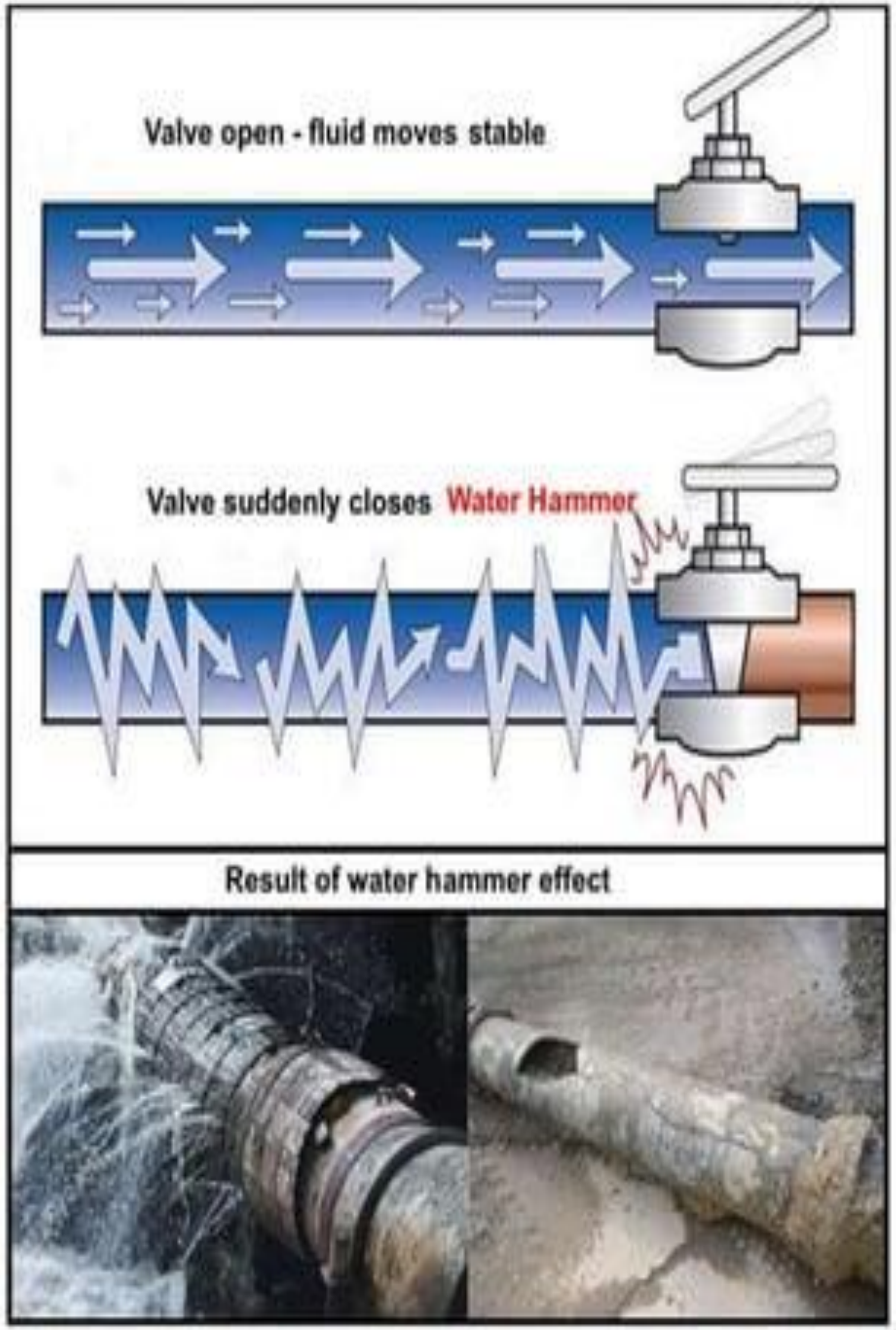
Unsteady flow is that type of flow in which fluid parameters (velocity, pressure, density etc.) at a point changes with time.

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} \neq 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0 \text{ etc.}$$

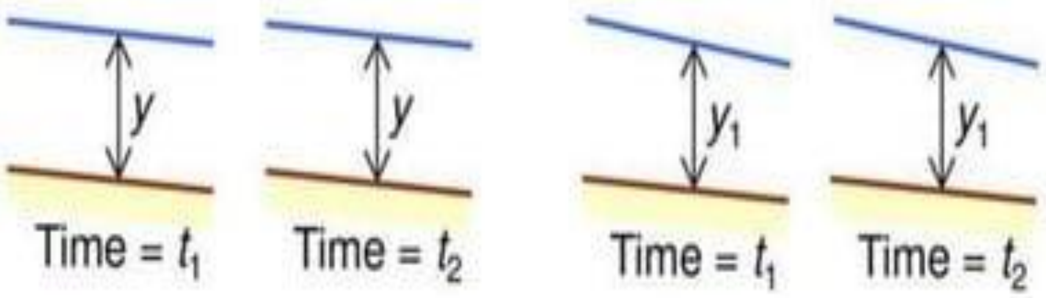


A flow is said to be **quasi steady** if temporal variations at a spatial location are much smaller (they would be zero if the flow was steady) compared to spatial variations for any quantity.

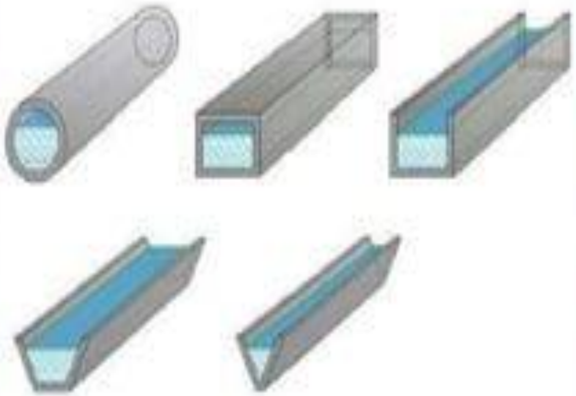



Valve open - fluid moves stable
Valve suddenly closes
Water Hammer
Result of water hammer effect

Steady flow



Unsteady flow



Uniform and Non-uniform Flows.

Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (i.e., length of direction of the flow). Mathematically, for uniform flow

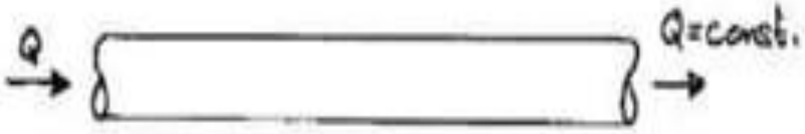
$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} = 0$$

where ∂V = Change of velocity

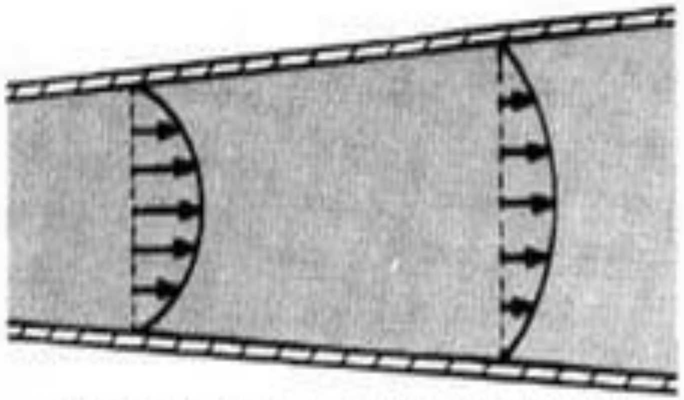
∂s = Length of flow in the direction S.

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

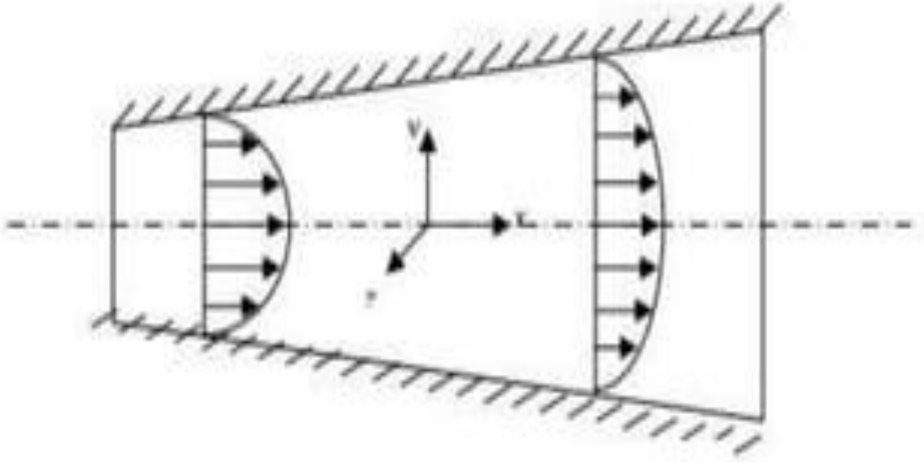
$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} \neq 0.$$



Steady uniform flow



Steady non-uniform flow



Three dimensional flow

Laminar flow is also called streamline or viscous flow. This type of flow occurs in smooth pipes having the low velocity of flow. It also occurs in liquids having high viscosity.

Turbulent flow is defined as that type of flow in which each fluid particle does not have a definite path and the paths of individual particles cross each other.

In other words, it is the flow in which fluid particles move in a zigzag path

When a fluid is flowing in a pipe, the type of flow is determined by a non-dimensional number, called Reynold's number.

For laminar flow, Reynold number < 2000

For turbulent flow Reynold number > 4000


Uniform and Non-Uniform Flow
Uniform flow
Non-uniform flow
Flow direction
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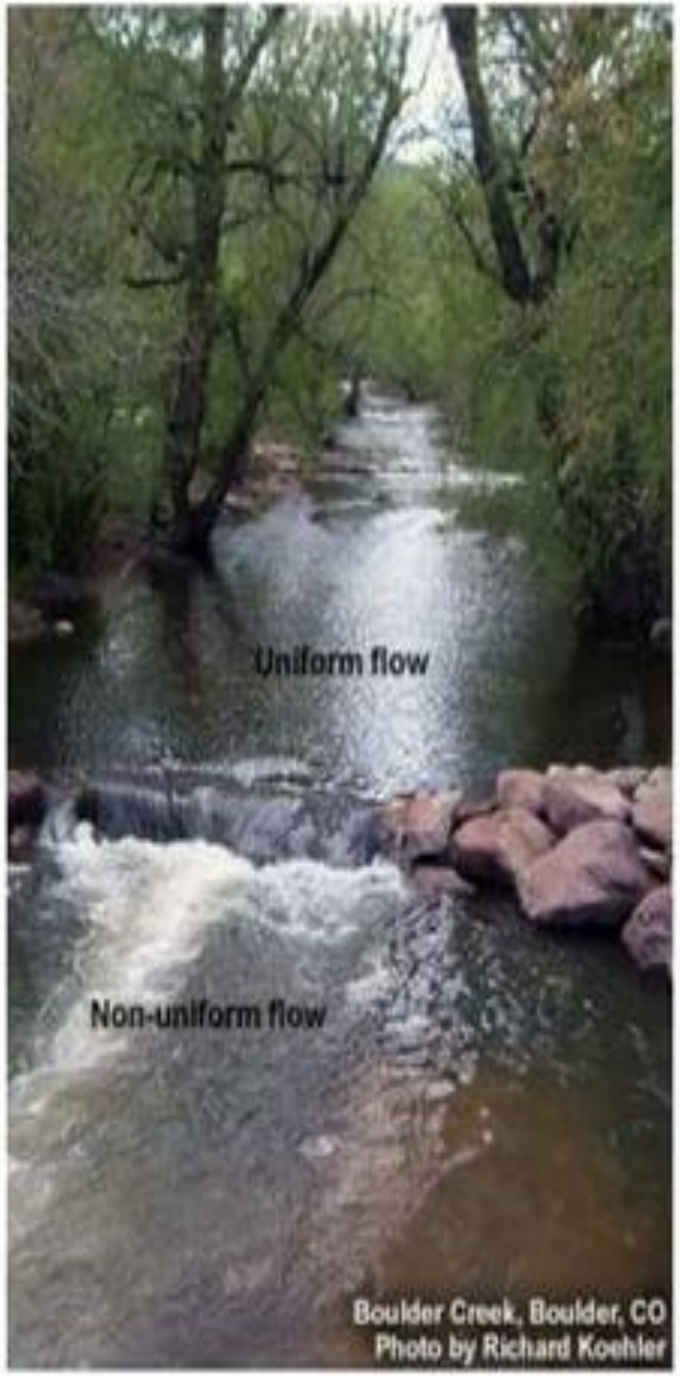
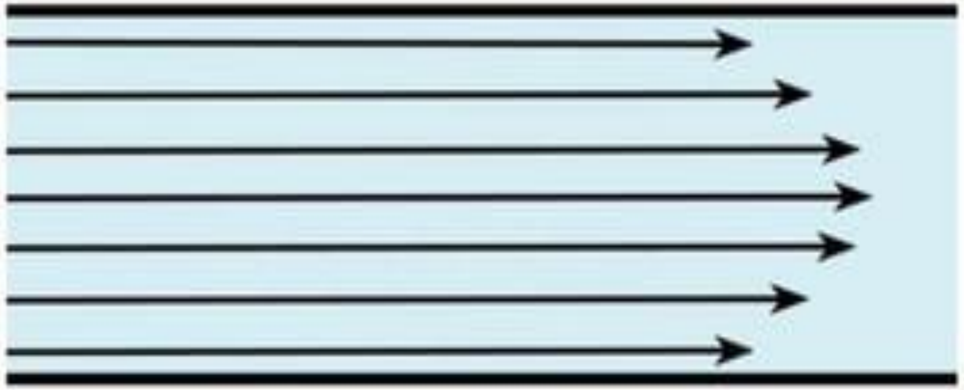
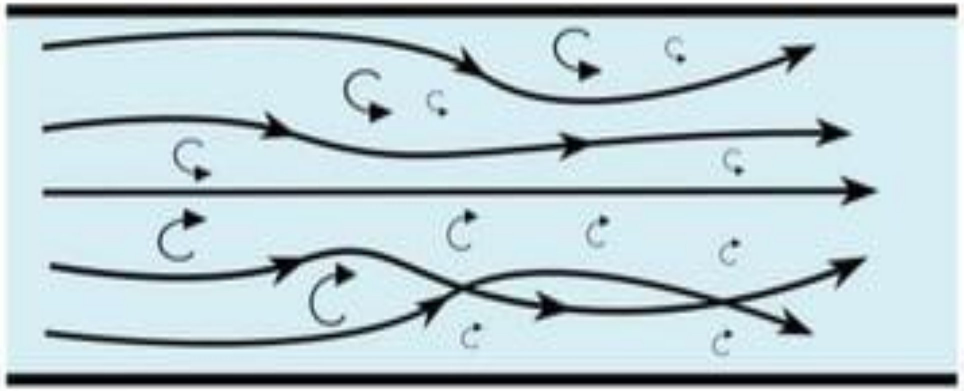
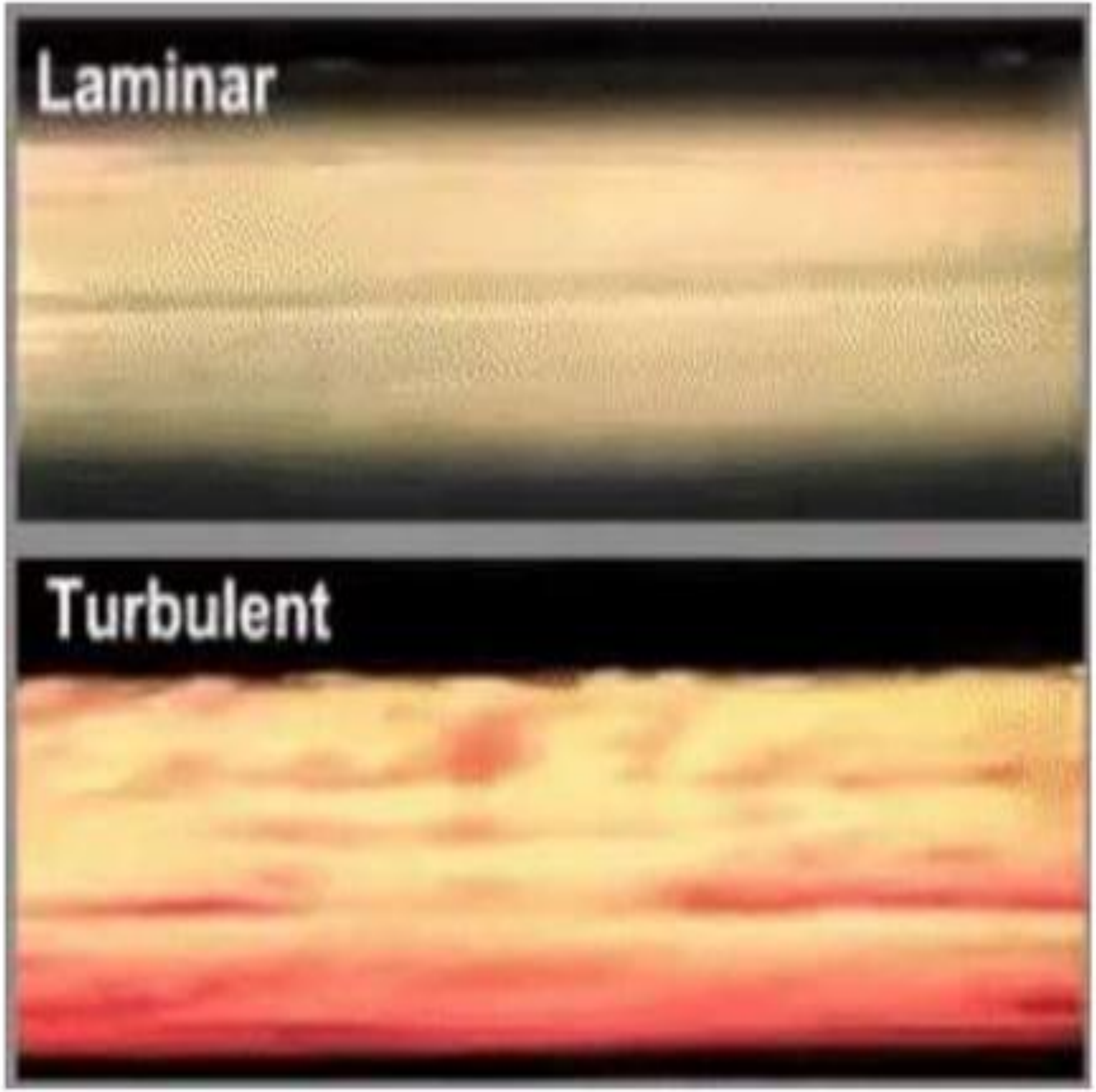

Uniform flow
Non-uniform flow
Boulder Creek, Boulder, CO
Photo by Richard Koehler

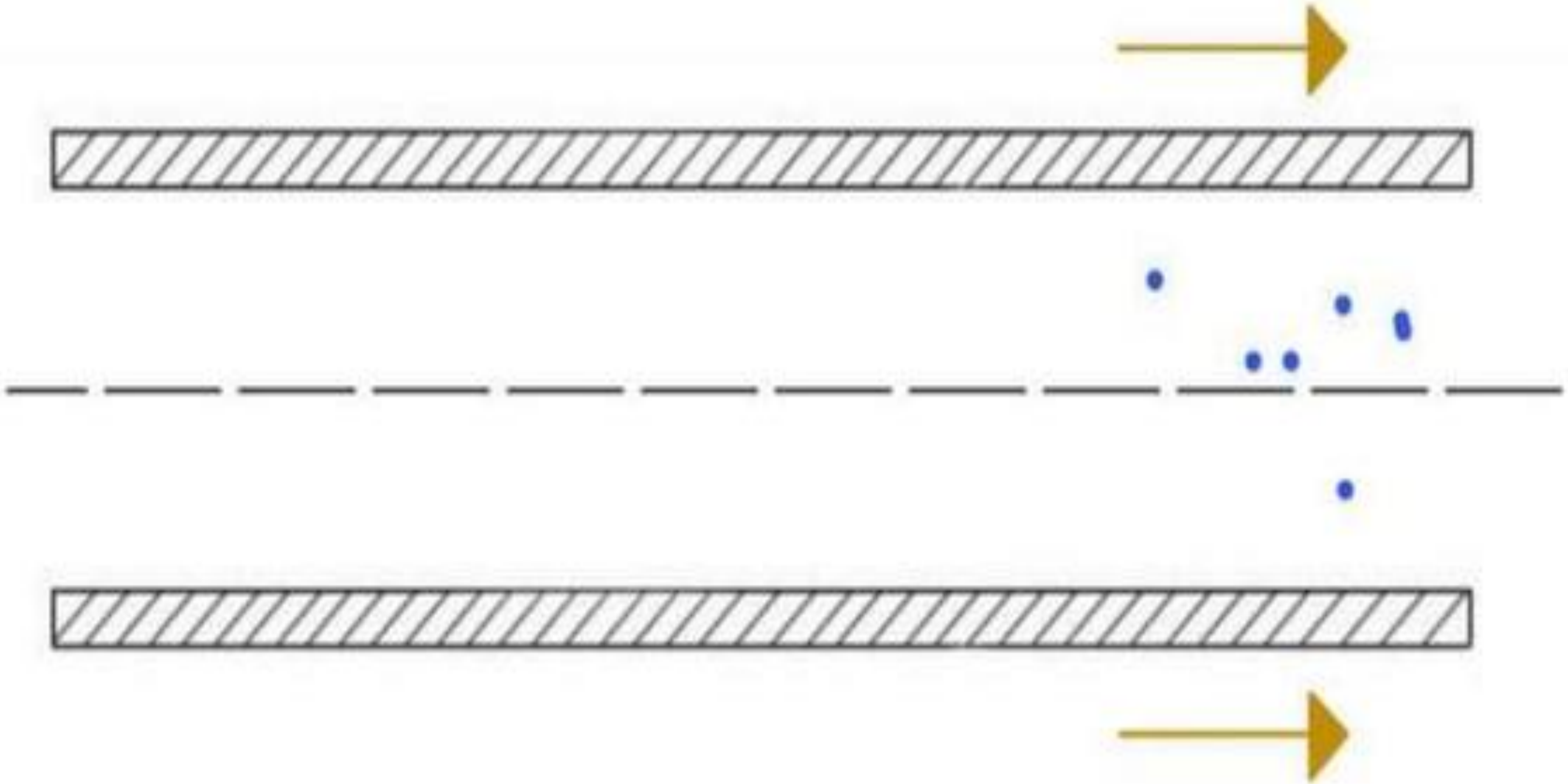
laminar flow



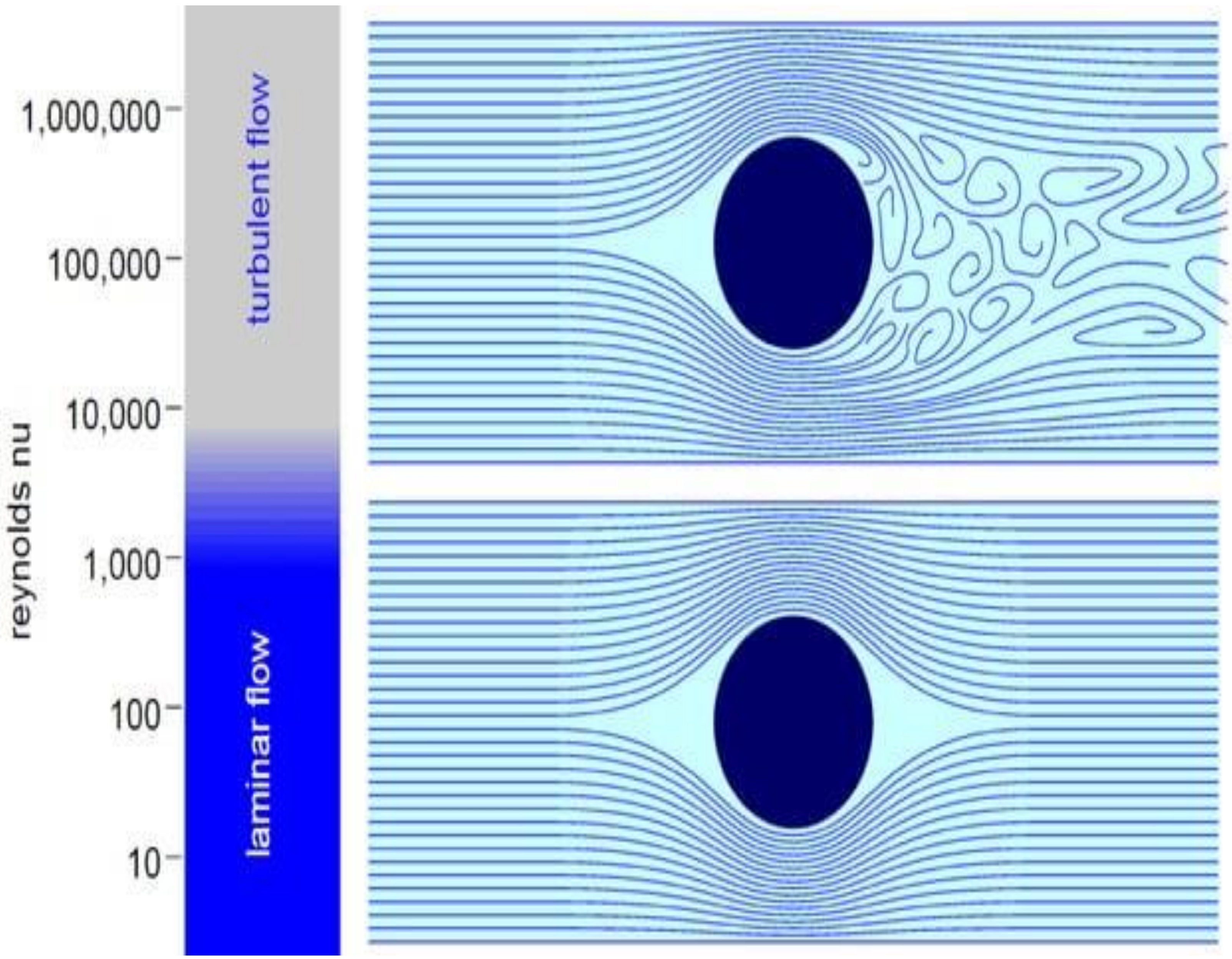
turbulent flow




Laminar
Turbulent



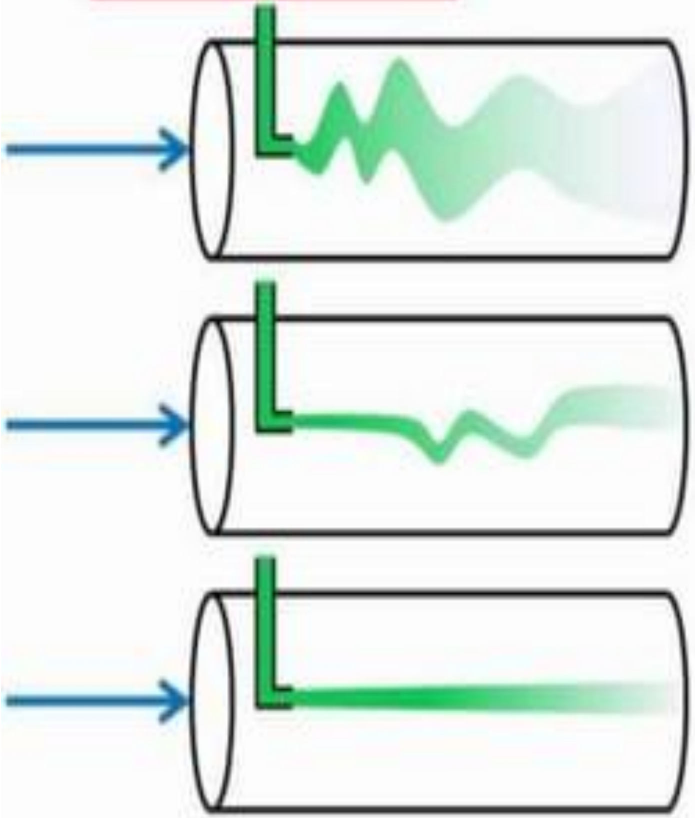
turbulent flow



1,000,000
100,000
10,000
1,000
100
10
reynolds nu
turbulent flow
laminar flow

The Reynolds number correlates well with flow characteristics.

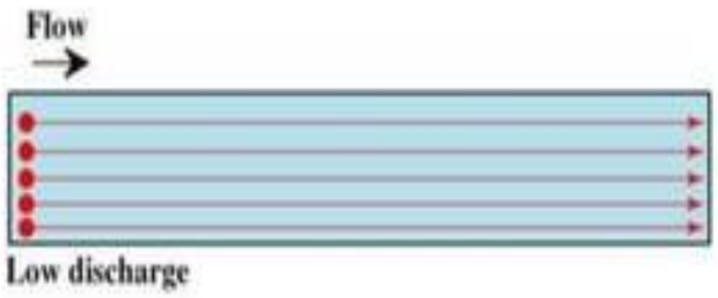
$$\mathrm{Re} = \frac{\rho V_{\mathrm{avg}} D}{\mu}$$



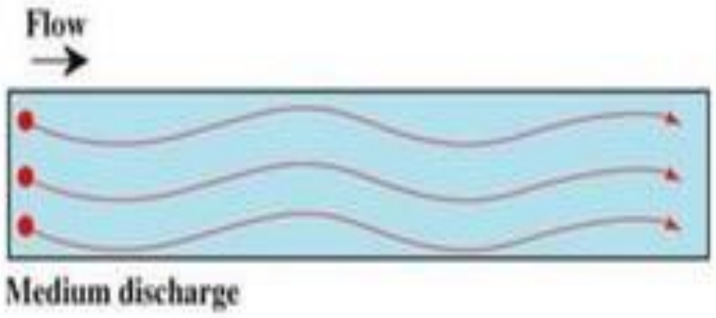
$\mathrm{Re} > 4000$
turbulent (unpredictable, rapid mixing)

$2300 < \mathrm{Re} < 4000$
transitional (turbulent outbursts)

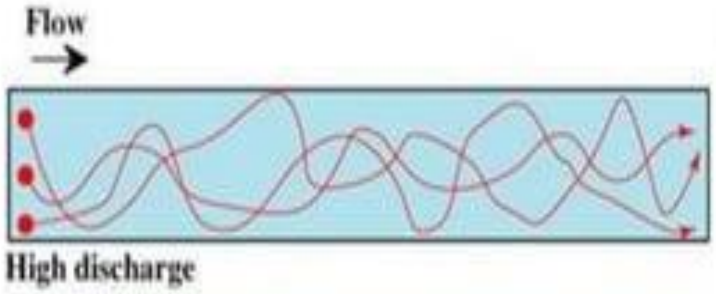
$\mathrm{Re} < 2300$
laminar (predictable, slow mixing)



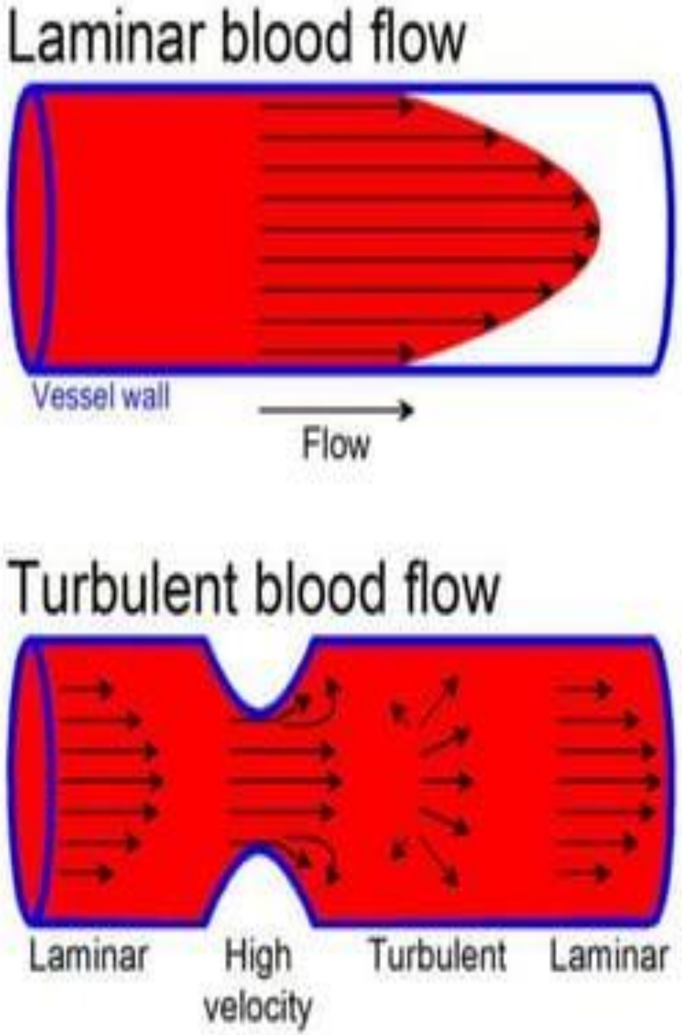
Laminar Flow: every fluid molecule followed a straight path that was parallel to the boundaries of the tube.



Transitional Flow: every fluid molecule followed wavy but parallel path that was not parallel to the boundaries of the tube.



Turbulent Flow: **E**very fluid molecule followed very complex path that led to a mixing of the dye.


Laminar blood flow
Vessel wall
Flow
Turbulent blood flow
Laminar
High velocity
Turbulent
Laminar

compressible flow: The flow in which the density of fluid changes, due to pressure and temperature variations, from point to point during the flow is called compressible flow.

In other words, it is the flow in which the density of a fluid is not constant during the flow.

Mathematically, for compressible flow,

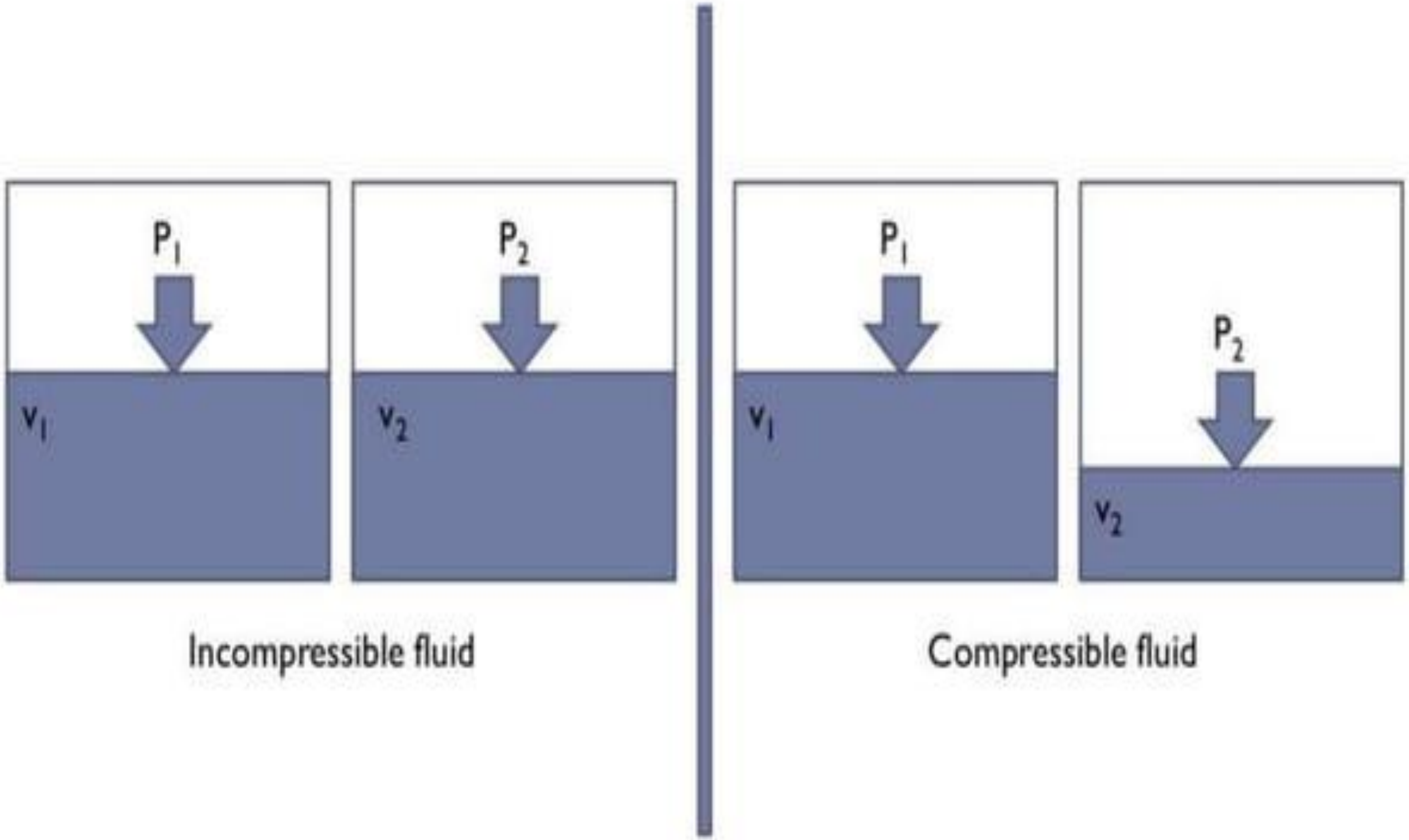
$$\rho \neq \text{constant}$$

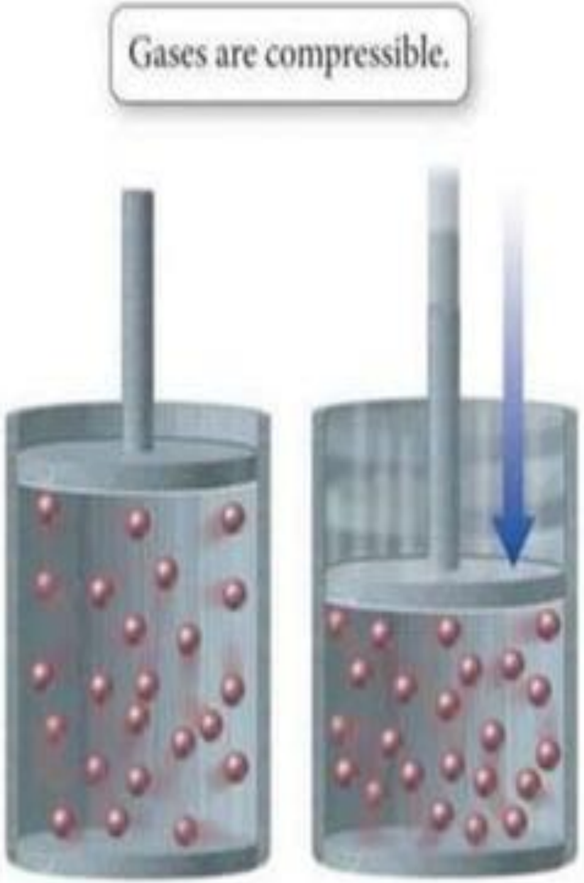
incompressible flow : The flow in which the density of fluid does not change during the flow is called incompressible flow. In other words, it is the flow in which the density of a fluid is constant during the flow.

Mathematically, for incompressible flow,

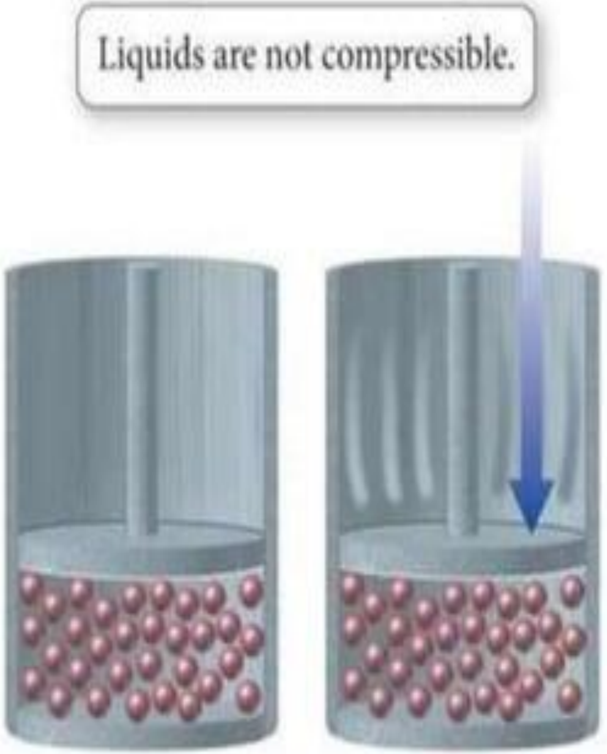
$$p = \text{constant}$$

Liquids are generally incompressible which means that pressure and temperature changes have a very little effect on their volume. Gases are compressible fluids.


P1
P2
v1
v2
P1
P2
v1
v2
Incompressible fluid
Compressible fluid



Gas

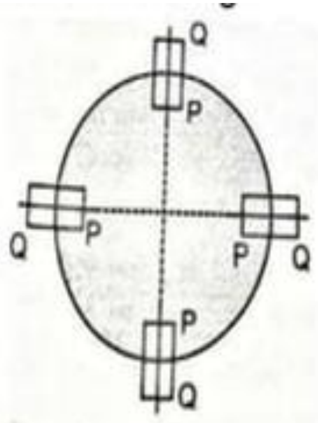


Liquid

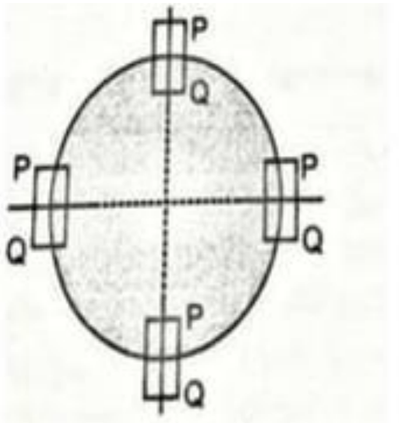
Compressible Flow

Rotational flow is that type of flow in which fluid particles also rotate about their own axes while flowing along a streamline.

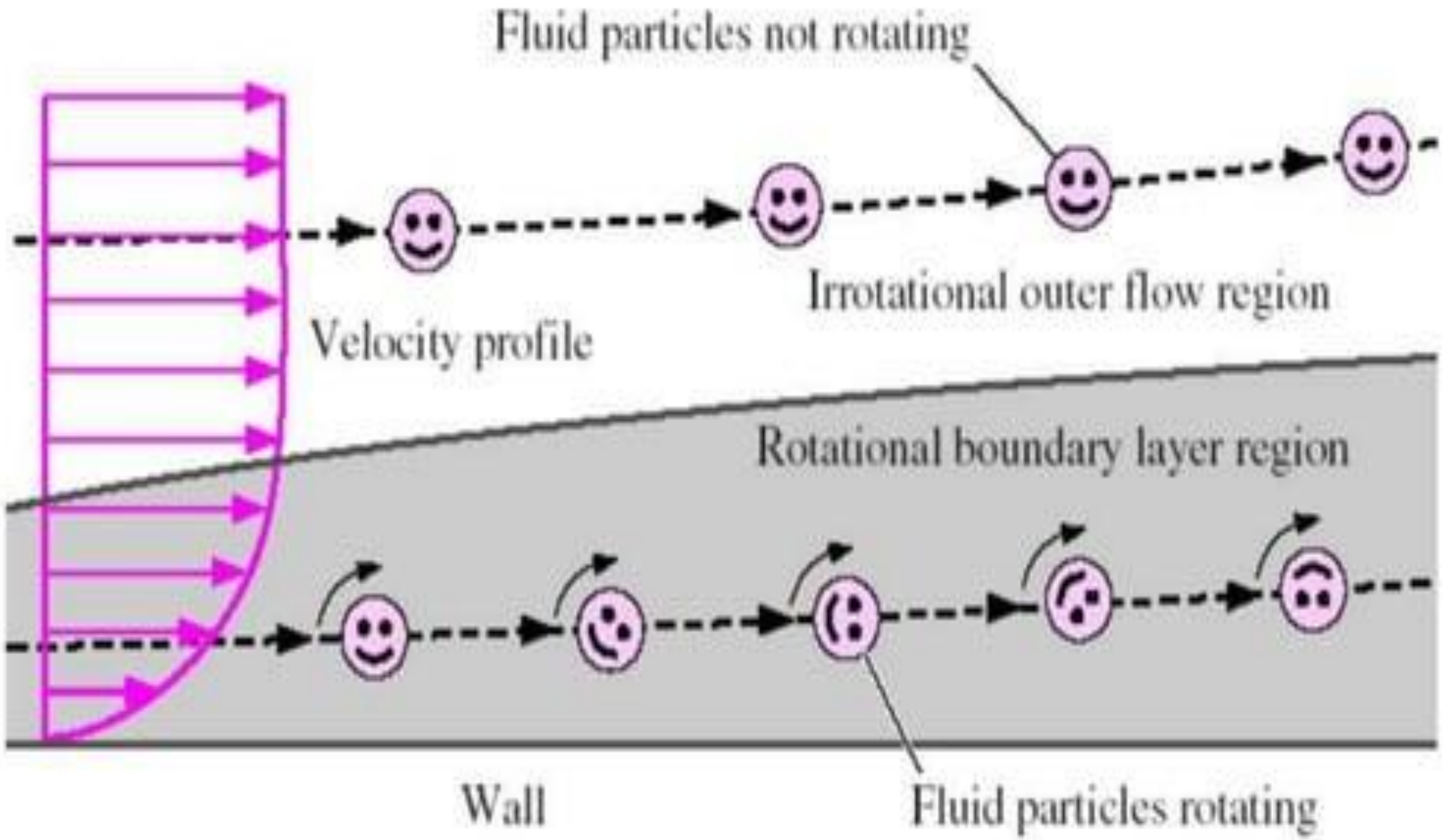
Irrotational flow is that type of flow in which fluid particles do not rotate about their own axes while flowing.



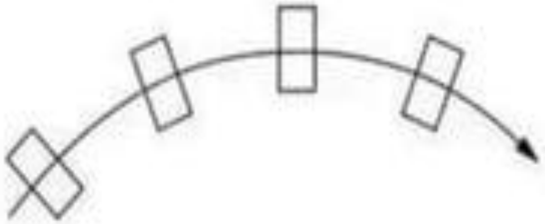
Rotational flow



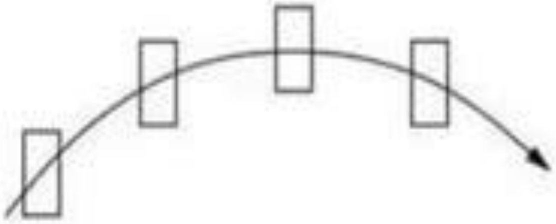
Irrotational flow



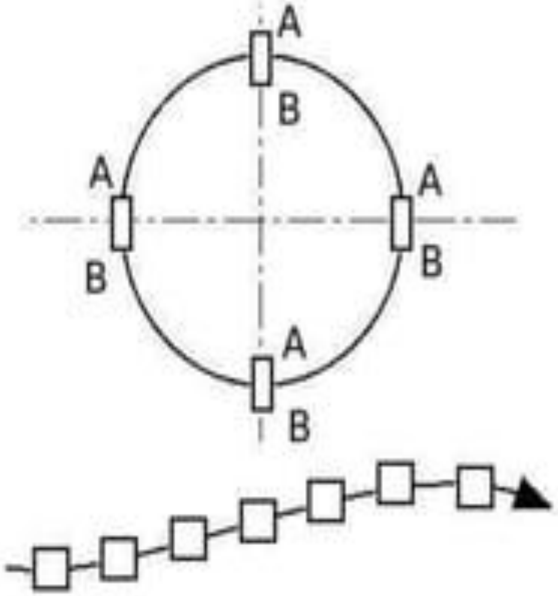
Fluid particles not rotating
Irrotational outer flow region
Velocity profile
Rotational boundary layer region
Wall
Fluid particles rotating



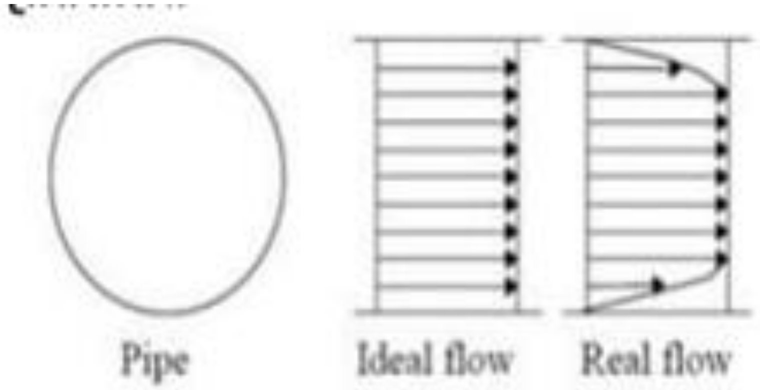
Rotational flows



Irrotational flows

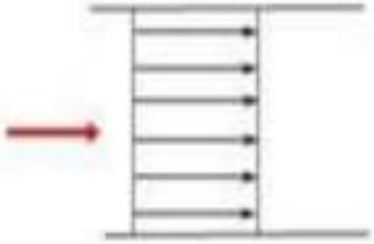


- An **ideal flow** is the flow of a non-viscous fluid. In the ideal flow, **no shear stress** exists between two adjacent layers or between the fluid layer and boundary, only normal stresses can exist in ideal flows.
- The **flow of real (viscous) fluids** is called real flow. In real flow, shear stress exists between to adjacent fluid layers. These stresses oppose the sliding of one layer over another



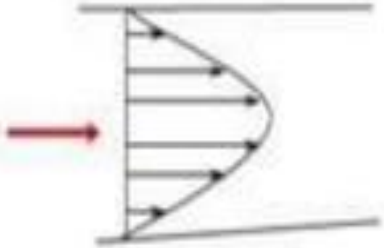
Velocity distribution of pipe flow

Real fluid flows implies friction effects. Ideal fluid flow is hypothetical; it assumes no friction.



Ideal

Friction = 0
Ideal Flow ($\mu = 0$)
Energy loss =0



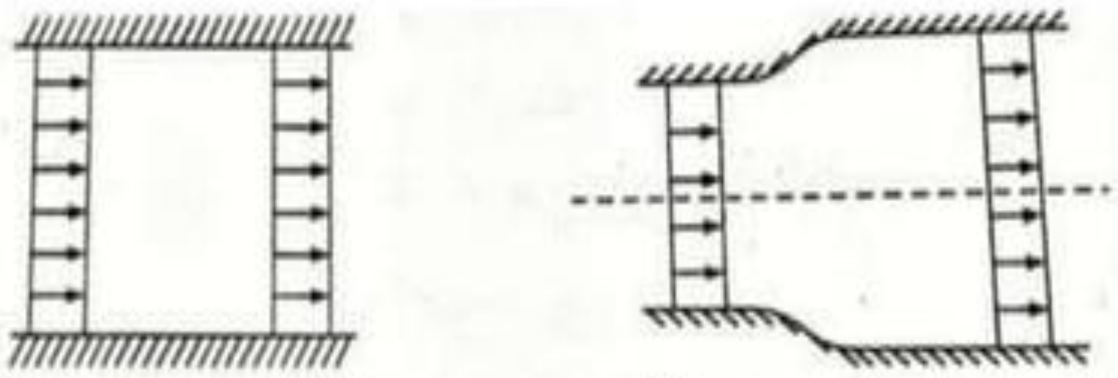
Real

Friction ≠ o
Real Flow ($\mu \neq 0$)
Energy loss ≠ 0

One dimensional flow is the flow in which parameters (velocity, pressure, density, viscosity and temperature) vary only in one direction and the flow is a function of only one co-ordinate Axis and time. The flow field is represented by streamlines which are straight and parallel

mathematically, for one-dimensional flow

$$u = f(x),\ v = 0 \text{ and } w = 0$$



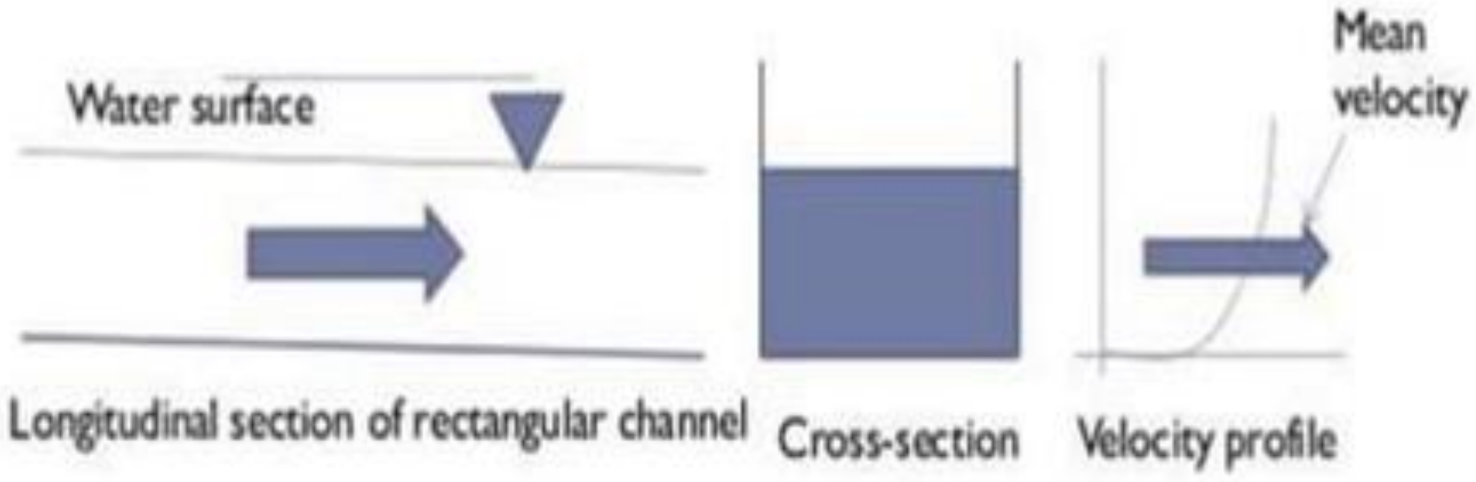
(a) Flow between parallel plates

(b) incompressible flow in a duct

Illustration of one dimensional flow

Although in general **all fluids flow three-dimensionally**, with pressures and velocities and other flow properties varying in all directions, in many cases the greatest changes only occur in two directions or even only in one. In these cases changes in the other direction can be effectively ignored making analysis much more simple.

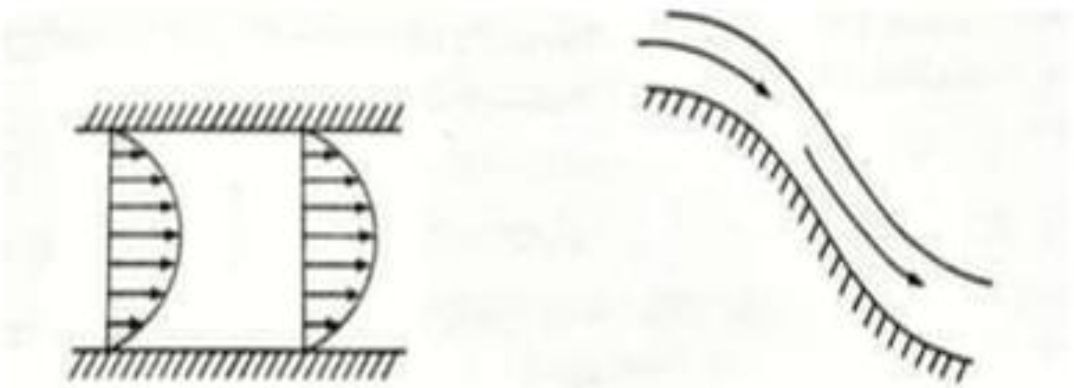
Flow is one dimensional if the flow parameters (such as velocity, pressure, depth etc.) at a given instant in time only vary in the direction of flow and not across the cross-section



Two-dimensional flow is the flow in which fluid parameters vary along two directions and the flow is the function of two rectangular space coordinates (x and y- axis) and time.

The flow field is represented by streamlines which are curves. mathematically for two-dimensional flow

$$u = f_1(x, y),\ v = f_2(x, y) \text{ and } w = 0.$$



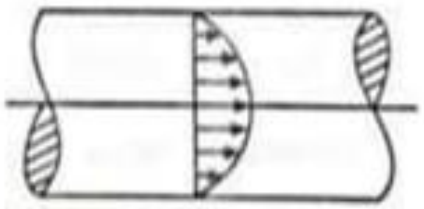
(a) Viscous flow between parallel plates

(b) Flow over a spillway

Illustration of two dimensional flow

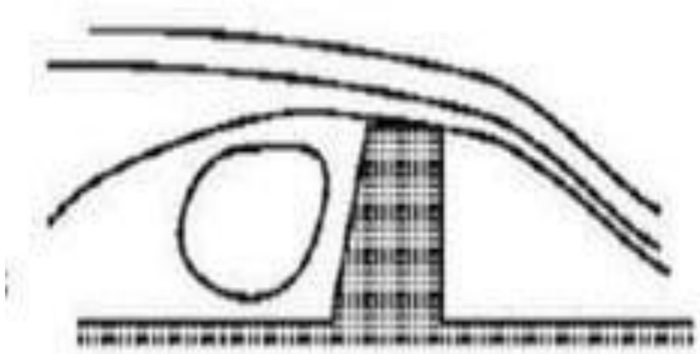
Three-dimensional flow is the flow in which flow parameters change in all the three directions and the flow is the functions of three mutually perpendicular co-ordinate Axis (x, y, z-axis) and time. The streamlines are space curves

$$u = f_1(x, y, z),\ v = f_2(x, y, z) \text{ and } w = f_3(x, y, z).$$



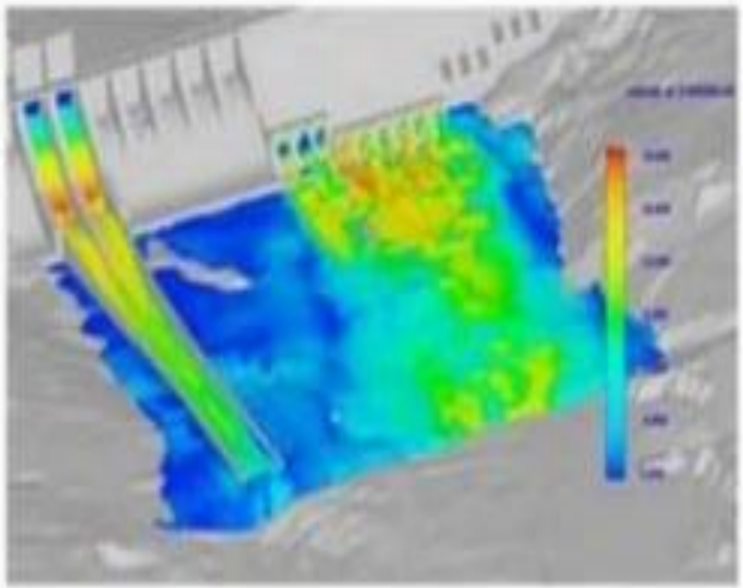
Viscous flow in a duct (three dimensional flow)

- Flow is two-dimensional if it can be assumed that the flow parameters vary in the direction of flow and in one direction at right angles to this direction



Two-dimensional flow over a weir

Flow is three-dimensional if the flow parameters vary in all three directions of flow



Three-dimensional flow in stilling basin

Rate of Flow or Flow Rate or Discharge:

Quantity of fluid passing through any section in a unit time.

Unit: m^3/sec

Types:

1. Volume Flow Rate = Volume of fluid / Time

   = Q or AV

2. Mass Flow Rate = Mass of fluid / Time = ρ x Volume / Time =

   ρ Q or PAV

3. Weigh Flow Rate = Weight of fluid / Time = (pg x Volume) / Time

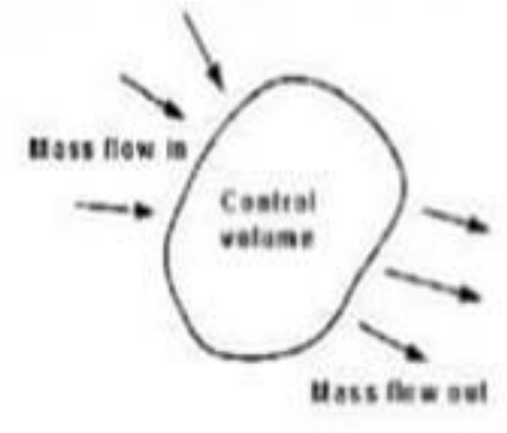
   = pgQ or pgAV   Where, Q = Area x Velocity

Continuity

- Matter cannot be created or destroyed

-(it is simply changed in to a different form of matter).

- This principle is know as the conservation of mass and we use it in the analysis of flowing fluids.
- The principle is applied to fixed volumes, known as control volumes shown in figure:
- For any control volume the principle of conservation of mass says



An arbitrarily shaped control volume.

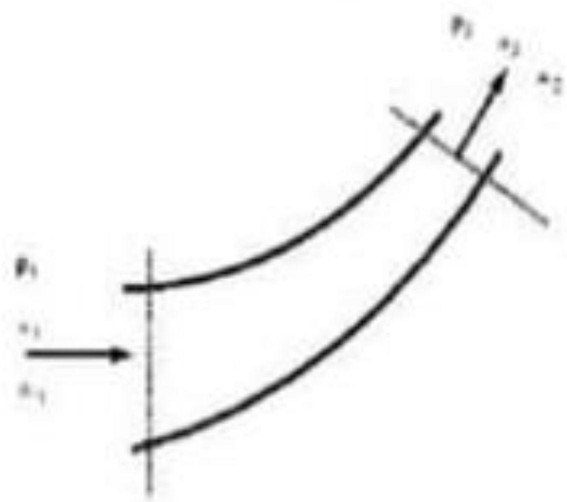
Mass entering per unit time - Mass leaving per unit time = Increase of mass in the control volume per unit time

Continuity Equation

- For steady flow there is no increase in the mass within the control volume, so Mass entering per unit time = Mass leaving per unit time
- **Derivation:**

Lets consider a stream tube.

$\rho_1 . V_1$ and A_1 are mass density. velocity and cross-sectional area at section 1. Similarly, $\rho_2 . V_2$ and A_2 are mass density, velocity and cross-sectional area at section 2.



A stream tube

$$M_1 = \rho_1 A_1 V_1$$
$$M_2 = \rho_2 A_2 V_2$$

- According to mass conservation

$$M_1 - M_2 = \frac{d(M_{CV})}{dt}$$
$$\rho_1 A_1 V_1 - \rho_2 A_2 V_2 = \frac{d(M_{CV})}{dt}$$

- For steady flow condition $d(M_{cv})/dt = 0$

$$\rho_1 A_1 V_1 - \rho_2 A_2 V_2 = 0 \Rightarrow \rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

$$M = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

- Hence, for stead flow condition, mass flow rate at section 1= mass flow rate at section 2. i.e., mass flow rate is constant.
- Similarly $G = \rho_1 g A_1 V_1 = \rho_2 g A_2 V_2$
- Assuming incompressible fluid, $\rho_1 = \rho_2 = \rho$

$$A_1 V_1 = A_2 V_2 \Rightarrow Q_1 = Q_2 \Rightarrow Q_1 = Q_2 = Q_3 = Q_4$$

Continuity Equation

- Therefore, according to mass conservation for steady flow of incompressible fluids volume flow rate remains same from section to section.

Problem 5.1 *The diameters of a pipe at the sections 1 and 2 are 10 cm and 15 cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 is 5 m/s. Determine also the velocity at section 2.*

Solution. Given :

At section 1, $D_1 = 10 \text{ cm} = 0.1 \text{ m}$

$$A_1 = \frac{\pi}{4}(D_1^2) = \frac{\pi}{4}(.1)^2 = 0.007854 \text{ m}^2$$

$V_1 = 5 \text{ m/s.}$

At section 2, $D_2 = 15 \text{ cm} = 0.15 \text{ m}$

$$A_2 = \frac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$$

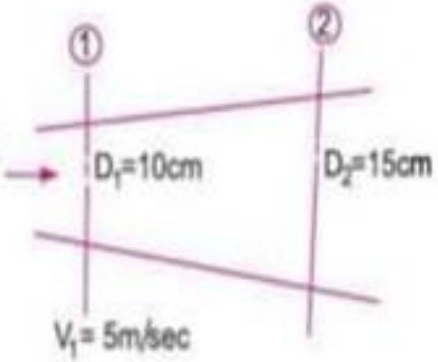


Fig. 5.2

(*i*) Discharge through pipe is given by equation (5.1)

or $Q = A_1 \times V_1$

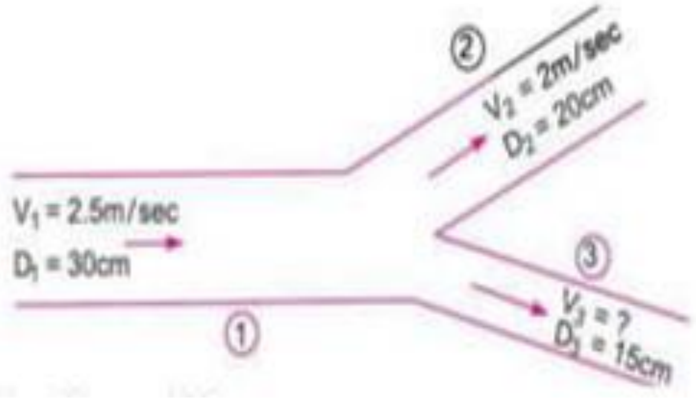
$= 0.007854 \times 5 = \mathbf{0.03927\ m^3/s.}$ **Ans.**

Using equation (5.3), we have $A_1V_1 = A_2V_2$

(*ii*) ∴ $V_2 = \frac{A_1V_1}{A_2} = \frac{0.007854}{0.01767} \times 5.0 = \mathbf{2.22\ m/s.}$ **Ans.**

Problem 5.2 *A 30 cm diameter pipe, conveying water, branches into two pipes of diameters 20 cm and 15 cm respectively. If the average velocity in the 30 cm diameter pipe is 2.5 m/s, find the discharge in this pipe. Also determine the velocity in 15 cm pipe if the average velocity in 20 cm diameter pipe is 2 m/s.*

Solution. Given :



$$D_1 = 30 \text{ cm} = 0.30 \text{ m}$$

$$\therefore \quad A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times .3^2 = 0.07068 \text{ m}^2$$

$$V_1 = 2.5 \text{ m/s}$$

$$D_2 = 20 \text{ cm} = 0.20 \text{ m}$$

$$\therefore \quad A_2 = \frac{\pi}{4} (.2)^2 = \frac{\pi}{4} \times .4 = 0.0314 \text{ m}^2,$$

$$V_2 = 2 \text{ m/s}$$

$$D_3 = 15 \text{ cm} = 0.15 \text{ m}$$

$$\therefore \quad A_3 = \frac{\pi}{4} (.15)^2 = \frac{\pi}{4} \times 0.225 = 0.01767 \text{ m}^2$$

Find (*i*) Discharge in pipe 1 or Q_1

(*ii*) Velocity in pipe of dia. 15 cm or V_3

Let Q_1, Q_2 and Q_3 are discharges in pipe 1, 2 and 3 respectively.

Then according to continuity equation

$$Q_1 = Q_2 + Q_3 \quad \text{...(1)}$$

(*i*) **The discharge $\mathbf{Q_1}$** in pipe 1 is given by

$$Q_1 = A_1 V_1 = 0.07068 \times 2.5 \text{ m}^3\text{/s} = \mathbf{0.1767 \text{ m}^3\text{/s. Ans.}}$$

(*ii*) **Value of $\mathbf{V_3}$**

$$Q_2 = A_2 V_2 = 0.0314 \times 2.0 = 0.0628 \text{ m}^3\text{/s}$$

Substituting the values of Q_1 and Q_2 in equation (1)

$$0.1767 = 0.0628 + Q_3$$

$\therefore$ $$Q_3 = 0.1767 - 0.0628 = 0.1139 \text{ m}^3\text{/s}$$

But $$Q_3 = A_3 \times V_3 = 0.01767 \times V_3 \quad \text{or} \quad 0.1139 = 0.01767 \times V_3$$

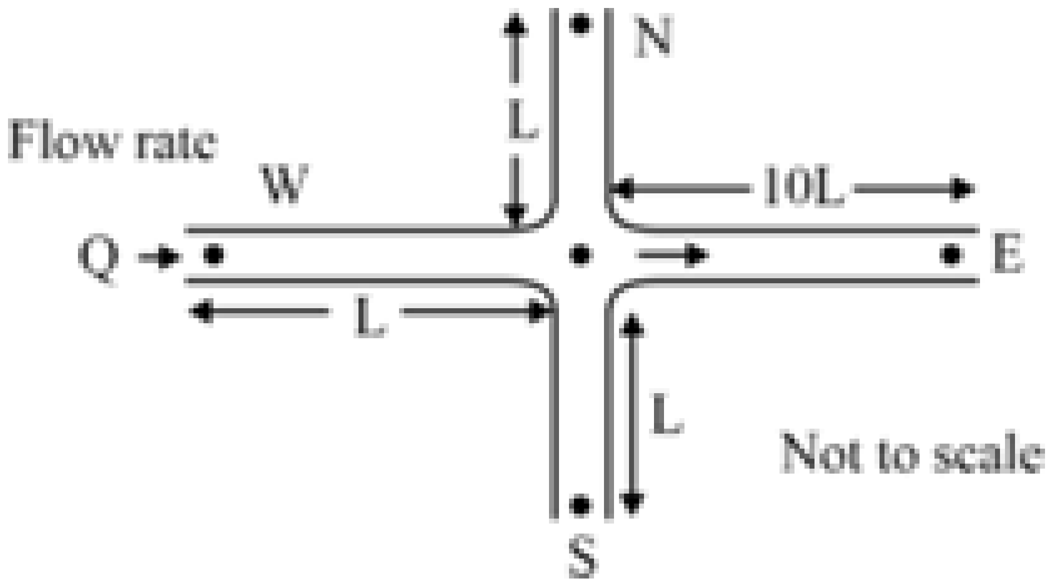
$\therefore$ $$V_3 = \frac{0.1139}{0.01767} = \mathbf{6.44 \text{ m/s. Ans.}}$$

[MCQ-2]

Q.33. In the pipe network as shown in figure, all pipes have the same cross section areas and can be assumed to have the same friction factor. The pipes connecting points W, N and S with the joint J have an equal length L. The pipe connecting points J and E has a length 10L. The pressures at the ends N, E and S are equal. The flow rate in the pipe connecting W and J is Q. Assume that the fluid flow is steady, incompressible, and the pressure losses at the pipe entrance and the junction are negligible, Consider the following statements.

I. The flow rate in pipe connecting J & E is $\frac{Q}{21}$.

II. The pressure difference between J & N is equal to the pressure difference between J & E.



Sol. **(d)**

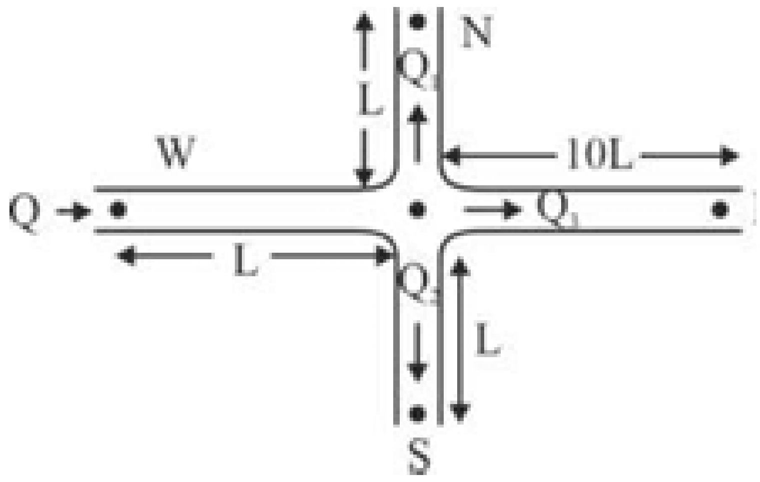
$Q_1 + Q_2 + Q_3 = Q$ .... (i)

$Q_1 = Q_2$ .... (ii)

From equation (i) and (ii)

$2Q_1 + Q_3 = Q$ ....(A)

$P_J - P_N = P_J - P_E$



Now $\frac{P_J - P_N}{\rho g} = \frac{P_J - P_E}{\rho g}$

$(h_L)_{JN} = (h_L)_{JE}$

$\frac{fL_{JN}Q_1^2}{12.1\,D^5} = \frac{fL_{JE}Q_3^2}{12.1\,D^5}$

$LQ_1^2 = 10LQ_3^2$

$Q_1 = \sqrt{10}Q_3$

From equation (A)

$2Q_1 + Q_3 = Q$

$2\sqrt{10}Q_3 + Q_3 = Q$

$Q_3 = \frac{Q}{1+2\sqrt{10}}$

$Q_3 = 0.136\ Q$

So statement 1 is wrong and statement 2 is correct.

Velocity of Fluid Particle

Velocity of a fluid along any direction can be defined as the rate of change of displacement of the fluid along that direction

Resultant velocity $\bar{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$$|V| = \sqrt{u^2 + v^2 + w^2}$$

Acceleration of Fluid Particle

• Acceleration of a fluid element along any direction can be defined as the rate of change of velocity of the fluid along that direction.

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$|a| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Where a_x, a_y, a_z are the acceleration vectors in X, Y and Z directions, respectively.

- a_x, a_y and a_z are termed as the total acceleration on respective directions.

- Total acceleration has two components – w.r.t time and w.r.t space and can be expressed in terms of u, v and w and this can be represented as:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

local acceleration
or
temporal acceleration

convective acceleration

- Euler equations are applicable to compressible, incompressible, non-viscous in steady or unsteady state of flow.

Let V is the resultant velocity at any point in a fluid flow. Let u, v and w are its component in x, y and z directions. The velocity components are functions of space-co-ordinates and time. Mathematically, the velocity components are given as

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

and Resultant velocity, $\quad V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x, a_y and a_z are the **total acceleration** in x, y and z directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But $\quad \frac{dx}{dt} = u, \frac{dy}{dt} = v \text{ and } \frac{dz}{dt} = w$

$\therefore$

$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Similarly,

$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \qquad ...(5.6)$$

$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Local Acceleration

For steady flow, $\frac{\partial V}{\partial t} = 0$, where V is resultant velocity

or $$\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0 \text{ and } \frac{\partial w}{\partial t} = 0$$

Hence acceleration in x, y and z directions becomes

$$\left.\begin{aligned} a_x &= \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right] \quad ...(5.7)$$

Acceleration vector $$\left.\begin{aligned} A &= a_x i + a_y j + a_z k \\ &= \sqrt{a_x^2 + a_y^2 + a_z^2}. \end{aligned}\right| \quad ...(5.8)$$

Problem 5.6 *The velocity vector in a fluid flow is given*

$$V = 4x^3 i - 10x^2 yj + 2tk.$$

Find the velocity and acceleration of a fluid particle at (2, 1, 3) at time t = 1.

Solution. The velocity components u, v and w are $u = 4x^3$, $v = -10x^2 y$, $w = 2t$

For the point (2, 1, 3), we have $x = 2$, $y = 1$ and $z = 3$ at time $t = 1$.

Hence velocity components at (2, 1, 3) are

$$u = 4 \times (2)^3 = 32 \text{ units}$$

$$v = -10(2)^2(1) = -40 \text{ units}$$

$$w = 2 \times 1 = 2 \text{ units}$$

$\therefore$ Velocity vector V at (2, 1, 3) $= 32i - 40j + 2k$

or Resultant velocity $= \sqrt{u^2 + v^2 + w^2}$

$$= \sqrt{32^2 + (-40)^2 + 2^2} = \sqrt{1024 + 1600 + 4} = \textbf{51.26 units. Ans.}$$

Acceleration is given by equation (5.6)

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Now from velocity components, we have

$$\frac{\partial u}{\partial x} = 12x^2,\ \frac{\partial u}{\partial y} = 0,\ \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = -20xy,\ \frac{\partial v}{\partial y} = -10x^2,\ \frac{\partial v}{\partial z} = 0\ \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = 0,\ \frac{\partial w}{\partial y} = 0,\ \frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 2.1$$

Substituting the values, the acceleration components at (2, 1, 3) at time $t = 1$ are

$$a_x = 4x^3 (12x^2) + (-10x^2y)(0) + 2t \times (0) + 0$$

$$= 48x^5 = 48 \times (2)^5 = 48 \times 32 = 1536 \text{ units}$$

$$a_y = 4x^3 (-20xy) + (-10x^2y)(-10x^2) + 2t(0) + 0$$

$$= -80x^4y + 100x^4y$$

$$= -80(2)^4(1) + 100(2)^4 \times 1 = -1280 + 1600 = 320 \text{ units.}$$

$$a_z = 4x^3(0) + (-10x^2y)(0) + (2t)(0) + 2.1 = 2.0 \text{ units}$$

∴ Acceleration is $A = a_x i + a_y j + a_z k =$ **1536i + 320j + 2k. Ans.**

or Resultant $A = \sqrt{(1536)^2 + (320)^2 + (2)^2}$ units

$$= \sqrt{2359296 + 102400 + 4} = \mathbf{1568.9 \text{ units. Ans.}}$$

Continuity Equation in Three Dimension :

When fluid flow through a full pipe, the volume of fluid entering in to the pipe must be equal to the volume of the fluid leaving the pipe, even if the diameter of the pipe vary.

Therefore we can define the continuity equation as the equation based on the principle of conservation of mass.

Therefore, for a flowing fluid through the pipe at every cross-section, the quantity of fluid per second will be constant.

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z directions respectively. Mass of fluid entering the face $ABCD$ per second

$$= \rho \times \text{Velocity in } x\text{-direction} \times \text{Area of } ABCD$$

$$= \rho \times u \times (dy \times dz)$$

Then mass of fluid leaving the face $EFGH$ per second $= \rho u\, dydz + \frac{\partial}{\partial x}(\rho u\, dydz)\, dx$

$\therefore$ Gain of mass in x-direction

$$= \text{Mass through } ABCD - \text{Mass through } EFGH \text{ per second}$$

$$= \rho u\, dydz - \rho u\, dydz - \frac{\partial}{\partial x}(\rho u\, dydz) dx$$

$$= -\frac{\partial}{\partial x}(\rho u\, dydz)\, dx$$

$$= -\frac{\partial}{\partial x}(\rho u)\, dx\, dydz \qquad \{\because dydz \text{ is constant}\}$$

Similarly, the net gain of mass in y-direction

$$= -\frac{\partial}{\partial y}(\rho v)\, dxdydz$$

and in z-direction

$$= -\frac{\partial}{\partial z}(\rho w)\, dxdydz$$

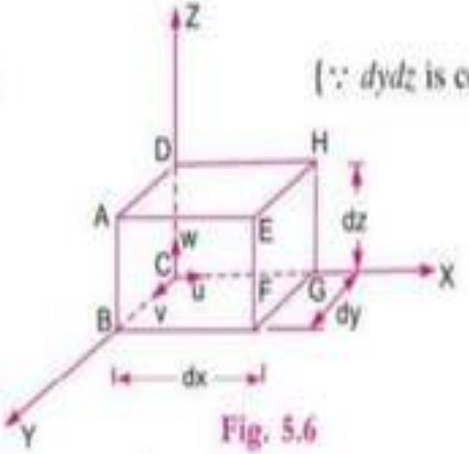


Fig. 5.6

$\therefore$ Net gain of masses $= -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass of fluid in the element is $\rho.\ dx.\ dy.\ dz$ and its rate of increase with time is $\frac{\partial}{\partial t}(\rho\ dx.\ dy.\ dz)$ or

$$\frac{\partial \rho}{\partial t}.\ dx\ dy\ dz.$$

Equating the two expressions,

or $$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t}.\ dxdydz$$

or $$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0 \text{ [Cancelling } dx.dy.dz \text{ from both sides]} \quad \ldots(5.3A)$$

Equation (5.3*A*) is the continuity equation in cartesian co-ordinates in its most general form. This equation is applicable to :

(*i*) Steady and unsteady flow,

(*ii*) Uniform and non-uniform flow, and

(*iii*) Compressible and incompressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.3*A*) becomes as

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad ...(5.3B)$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad ...(5.4)$$

Equation (5.4) is the continuity equation in three-dimensions. For a two-dimensional flow, the component $w = 0$ and hence continuity equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad ...(5.5)$$

Problem 5.7 *The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation :*

(i) $u = x^2 + y^2 + z^2$; $v = xy^2 - yz^2 + xy$

(ii) $v = 2y^2$, $w = 2xyz$.

Solution. The continuity equation for incompressible fluid is given by equation (5.4) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Case I. $u = x^2 + y^2 + z^2 \quad \therefore \quad \frac{\partial u}{\partial x} = 2x$

$$v = xy^2 - yz^2 + xy \quad \therefore \quad \frac{\partial v}{\partial y} = 2xy - z^2 + x$$

Substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ in continuity equation.

$$2x + 2xy - z^2 + x + \frac{\partial w}{\partial z} = 0$$

or
$$\frac{\partial w}{\partial z} = -3x - 2xy + z^2 \text{ or } \partial w = (-3x - 2xy + z^2)\,\partial z$$

Integration of both sides gives $\int dw = \int \left(-3x - 2xy + z^2\right) dz$

or $$w = \left(-3xz - 2xyz + \frac{z^3}{3}\right) + \text{Constant of integration},$$

where constant of integration cannot be a function of z. But it can be a function of x and y that is $f(x, y)$.

$$\therefore \quad w = \left(-3xz - 2xyz + \frac{z^3}{3}\right) + f(x, y). \textbf{ Ans.}$$

Case II. $$v = 2y^2 \quad \therefore \quad \frac{\partial v}{\partial y} = 4y$$

$$w = 2xyz \quad \therefore \quad \frac{\partial w}{\partial z} = 2xy$$

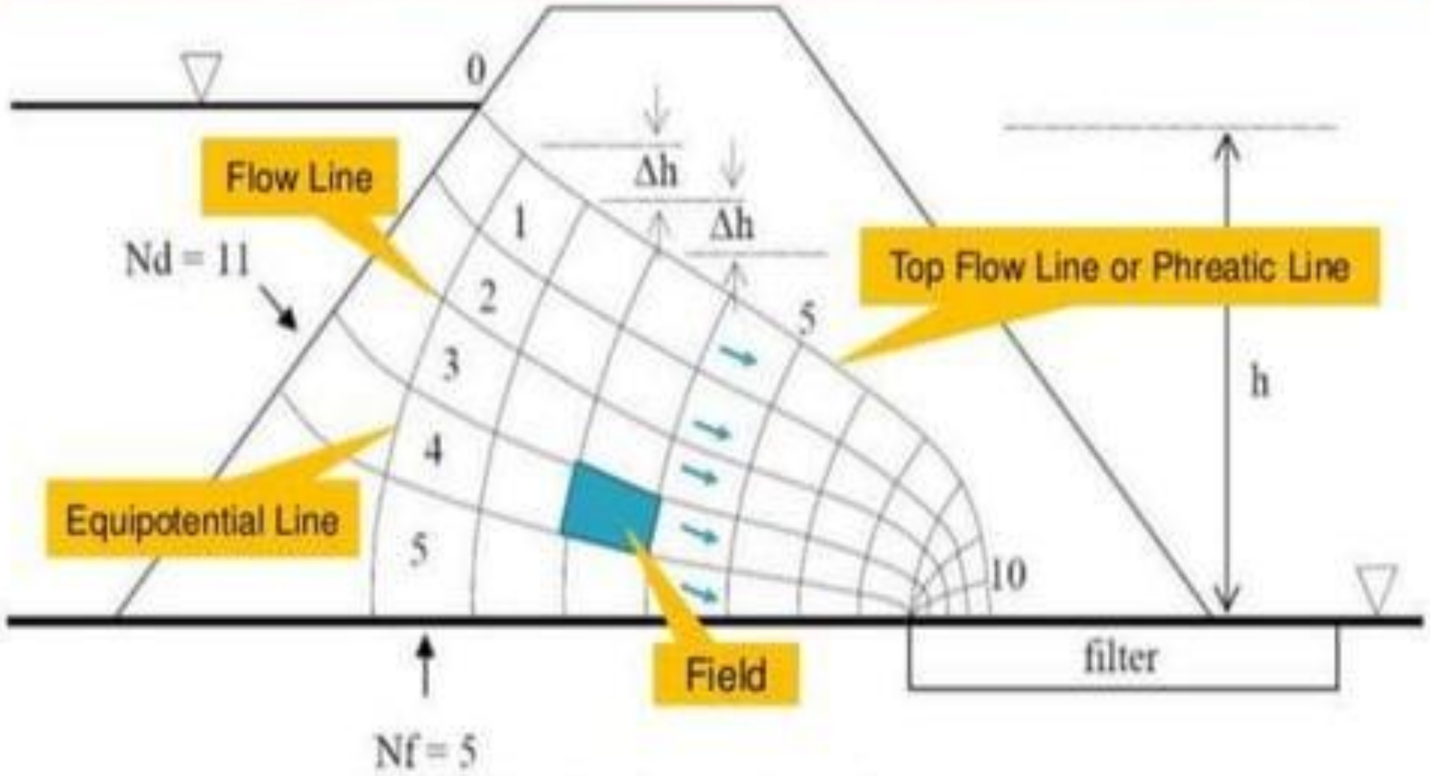
Substituting the values of $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ in continuity equation, we get

$$\frac{\partial u}{\partial x} + 4y + 2xy = 0$$

or $$\frac{\partial u}{\partial x} = -4y - 2xy \text{ or } du = (-4y - 2xy)\, dx$$

Integrating, we get $$u = -4xy - 2y\frac{x^2}{2} + f(y, z) = -4xy - x^2y + f(y, z). \textbf{ Ans.}$$

Phreatic Line is a seepage line separating saturated and unsaturated zones



Flow Net for an Earth Dam

Stream Function and Velocity Potential Function

5.8.1 Velocity Potential Function. It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi). Mathematically, the velocity, potential is defined as $\phi = f(x, y, z)$ for steady flow such that

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\} \qquad \text{...(5.9)}$$

where u, v and w are the components of velocity in x, y and z directions respectively.

The continuity equation for an incompressible steady flow is $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$

Substituting the values of u, v and w from equation (5.9), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right)=0$$

or

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0. \qquad ...(5.10)$$

Equation (5.10) is a Laplace equation.

For two-dimension case, equation (5.10) reduces to $\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0.$ ...(5.11)

If any value of ϕ that satisfies the Laplace equation, will correspond to some case of fluid flow.

Properties of the Potential Function. The rotational components* are given by

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

Substituting the values, of u, v and w from equation (5.9) in the above rotational components, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial x}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial z}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2\phi}{\partial z\partial x} + \frac{\partial^2\phi}{\partial x\partial z}\right]$$

and

$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial\phi}{\partial y}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2\phi}{\partial y\partial z} + \frac{\partial^2\phi}{\partial z\partial y}\right]$$

If ϕ is a continuous function, then $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$; $\frac{\partial^2\phi}{\partial z\partial x} = \frac{\partial^2\phi}{\partial x\partial z}$; etc.

$\therefore$ $$\omega_z = \omega_y = \omega_x = 0.$$

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are :

1. If velocity potential (ϕ) exists, the flow should be irrotational.

2. If velocity potential (ϕ) satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

Problem 5.11 *The velocity potential function is given by* $\phi = 5(x^2 - y^2)$. *Calculate the velocity components at the point (4, 5).*

Solution. $\phi = 5(x^2 - y^2)$

$$\therefore \quad \frac{\partial \phi}{\partial x} = 10x$$

$$\frac{\partial \phi}{\partial y} = -10y.$$

But velocity components u and v are given by equation (5.9) as

$$u = -\frac{\partial \phi}{\partial x} = -10x$$

$$v = -\frac{\partial \phi}{\partial y} = -(-10y) = 10y$$

The velocity components at the point (4, 5), *i.e.*, at $x = 4$, $y = 5$

$u = -10 \times 4 =$ **− 40 units. Ans.**

$v = 10 \times 5 =$ **50 units. Ans.**

5.8.2 Stream Function. It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (*Psi*) and defined only for two-dimensional flow. Mathematically, for steady flow it is defined as $\psi = f(x, y)$ such that

$$\left.\begin{aligned} \frac{\partial \psi}{\partial x} &= v \\ \text{and} \qquad \frac{\partial \psi}{\partial y} &= -u \end{aligned}\right\} \qquad \text{...(5.12)}$$

The velocity components in cylindrical polar co-ordinates in terms of stream function are given as

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \text{ and } u_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{...(5.12A)}$$

where u_r= radial velocity and u_θ = tangential velocity

The continuity equation for two-dimensional flow is $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0.$

Substituting the values of u and v from equation (5.12), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=0 \text{ or } -\frac{\partial^2 \psi}{\partial x \partial y}+\frac{\partial^2 \psi}{\partial x \partial y}=0.$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component ω_z is given by $\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$.

Substituting the values of u and v from equation (5.12) in the above rotational component, we get

$$\omega_z=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right]=\frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}\right]$$

For irrotational flow, $\omega_z = 0$. Hence above equation becomes as $\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial \psi}{\partial y^2}=0$

which is Laplace equation for ψ.

The **properties** of stream function (ψ) are :

1. If stream function (ψ) exists, it is a possible case of fluid flow which may be rotational or irrotational.

2. If stream function (ψ) satisfies the Laplace equation, it is a possible case of an irrotational flow.

5.8.6 Relation between Stream Function and Velocity Potential Function

From equation (5.9),

we have
$$u = -\frac{\partial \phi}{\partial x} \text{ and } v = -\frac{\partial \phi}{\partial y}$$

From equation (5.12), we have $u = -\dfrac{\partial \psi}{\partial y}$ and $v = \dfrac{\partial \psi}{\partial x}$

Thus, we have
$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

Hence
$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \text{and} \quad \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \quad \text{...(5.15)}$$

Problem 5.12 *A stream function is given by $\psi = 5x - 6y$.*
Calculate the velocity components and also magnitude and direction of the resultant velocity at any point.

Solution. $\psi = 5x - 6y$

$$\therefore \quad \frac{\partial \psi}{\partial x} = 5 \text{ and } \frac{\partial \psi}{\partial y} = -6.$$

But the velocity components u and v in terms of stream function are given by equation (5.12) as

$$u = -\frac{\partial \psi}{\partial y} = -(-6) = \mathbf{6\ units/sec.\ Ans.}$$

$$v = \frac{\partial \psi}{\partial x} = \mathbf{5\ units/sec.\ Ans.}$$

Resultant velocity $= \sqrt{u^2 + v^2} = \sqrt{6^2 + 5^2} = \sqrt{36 + 25} = \sqrt{61} = 7.81$ unit/sec

Direction is given by, $\tan\theta = \dfrac{v}{u} = \dfrac{5}{6} = 0.833$

$$\therefore \quad \theta = \tan^{-1} .833 = \mathbf{39^\circ\ 48'.\ Ans.}$$

Kinematics of flow

Kinematics of flow

INTRODUCTION

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics. Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined. In this chapter, the methods of determining velocity and acceleration are discussed.

Motion Characteristics:

- Velocity
- Acceleration
- Pressure
- Density

METHODS OF DESCRIBING FLUID MOTION

The fluid motion is described by two methods.

They are —(i) Lagrangian Method, and

(ii) Eulerian Method.

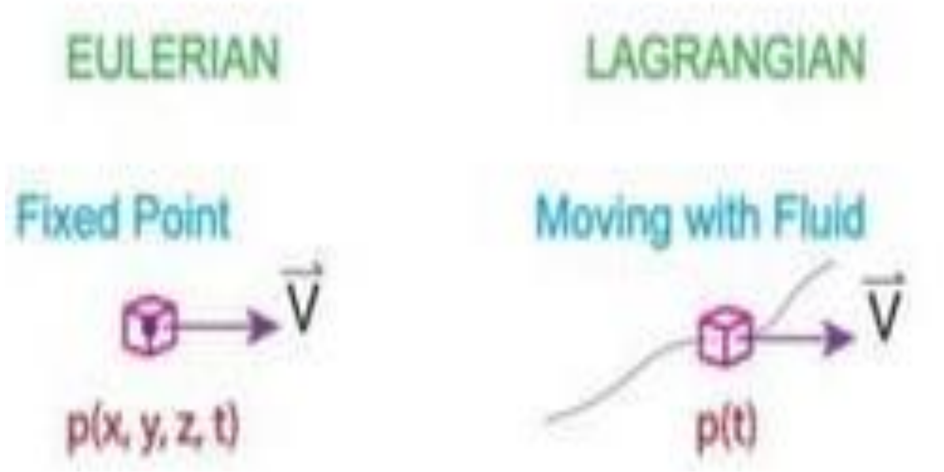
In the **Lagrangian method**, a single fluid particle is followed during its motion and its velocity, acceleration, density, etc., are described.

In case of **Eulerian method,** the velocity, acceleration, pressure, density etc., are described at a point in flow field. The Eulerian method is commonly used in fluid mechanics.

Fluid motion is described by two methods.

Methods:

Lagrangian Method - Describes a defined mass (position, velocity, acceleration, pressure, temperature etc.,) as functions of time.



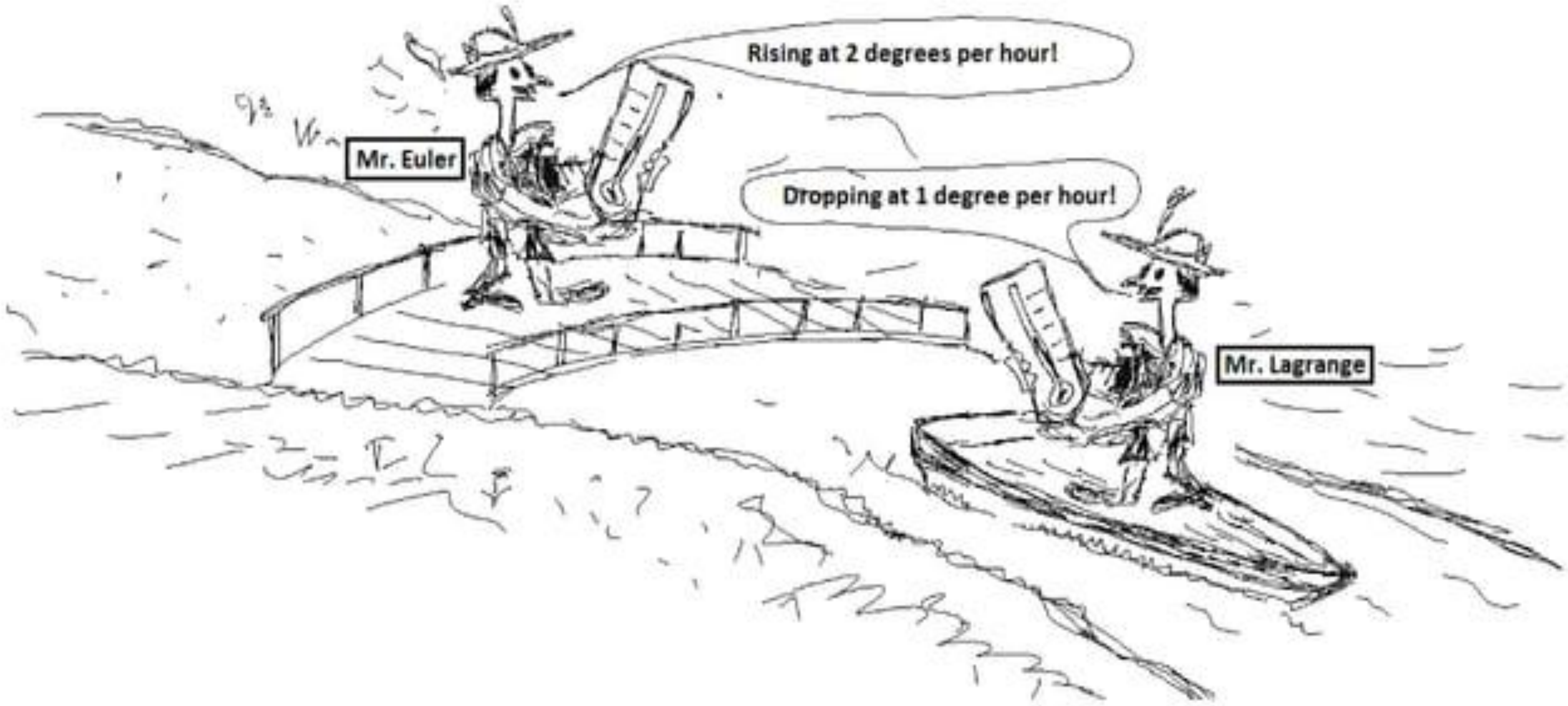
Example:

Track the location of a migrating bird

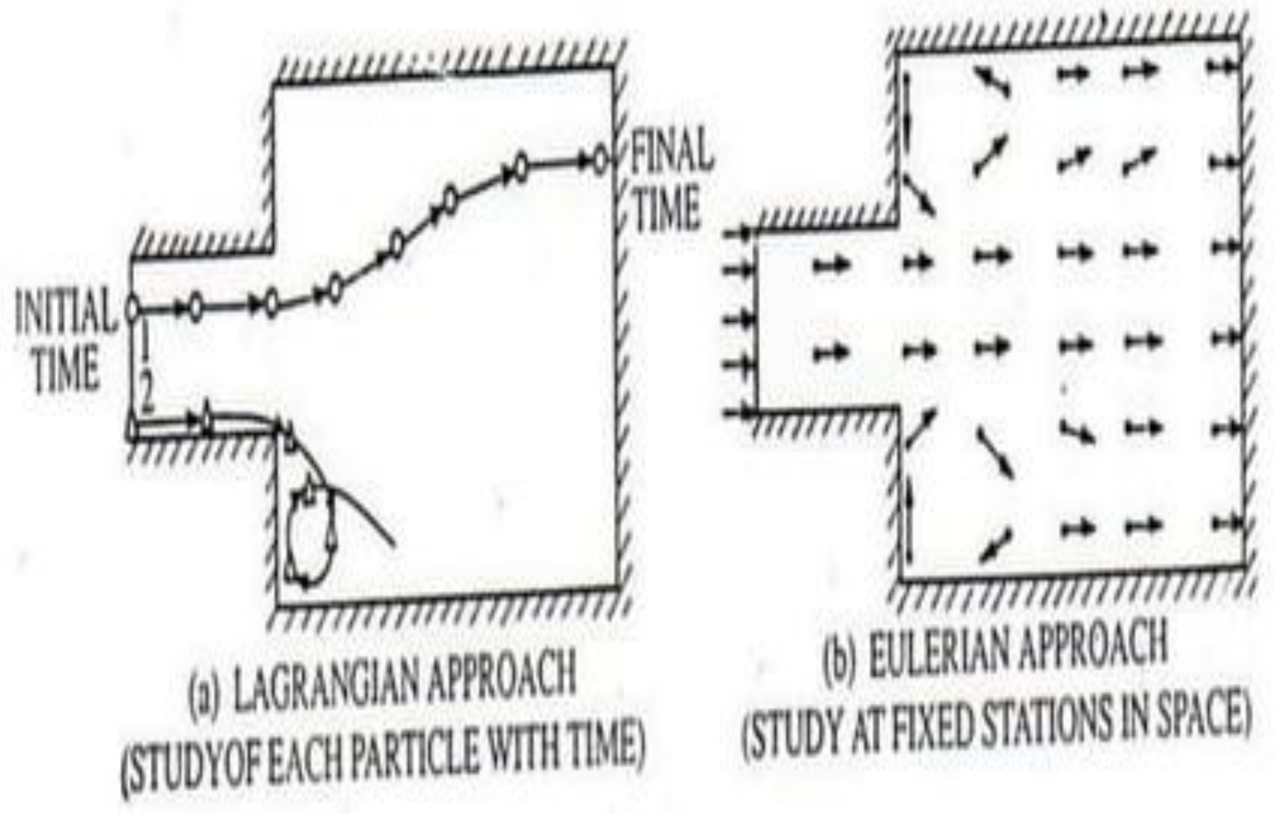
Eulerian Method - Describes a flow field (velocity, acceleration, pressure, temperature etc.,) as functions of position and time.

Example:

Count the birds passing through a particular location


Rising at 2 degrees per hour!
Dropping at 1 degree per hour!
Mr. Euler
Mr. Lagrange

- In the Eulerian approach, the fluid motion at all points in the flow field is determined by applying the laws of mechanics at all fixed stations.
- This is considerably easier than the Lagrangian approach and is followed in the study of Fluid Mechanics.



Difference Between Lagrangian and Eulerian Description

- Imagine a person standing beside a river measuring its properties.

In the Lagrangian approach, he throws in a probe that moves downstream with the water.

In the Eulerian approach, he anchors the probe at a fixed location in the water.

- **Experimental measurements** are more suitable to the Eulerian Description.

- Equations of motion of fluid flow in Lagrangian description are well defined (Newtons' second law), but needs to be carefully derived for the Eulerian description

Lagrangian method

- In this method, a single fluid particle is chosen and followed during its motion.
- Its velocity, acceleration, density etc. is described with respect to its location in space and time from a fixed position at the start of the motion.
- The position of the fluid particle (x,y,z) at any time t with respect to its position (a,b,c) at time t=0 is given as,

$x=f1(a,b,c,t)$; $y=f2(a,b,c,t)$; $z=f3(a,b,c,t)$

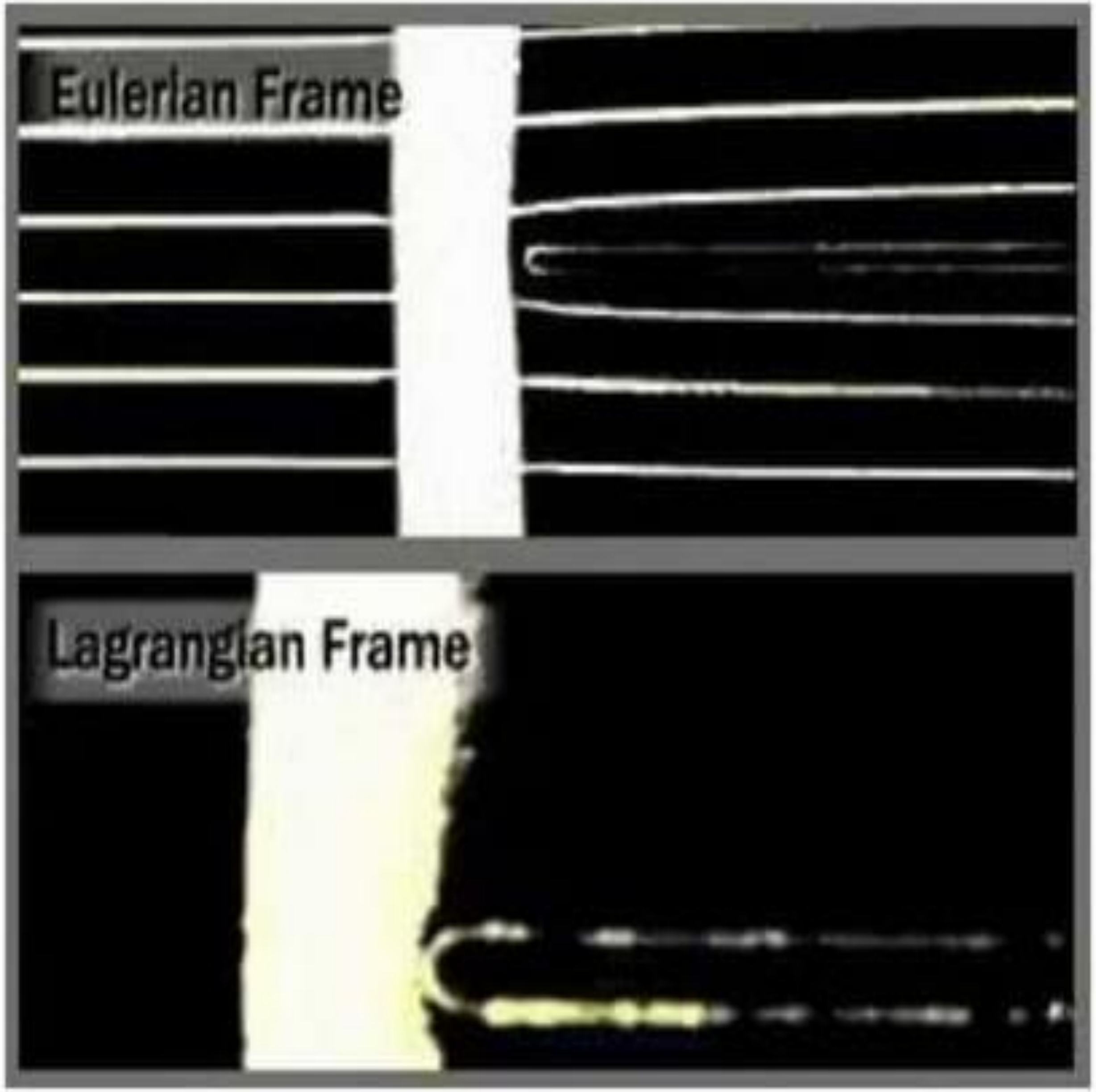
Then, velocity is given by

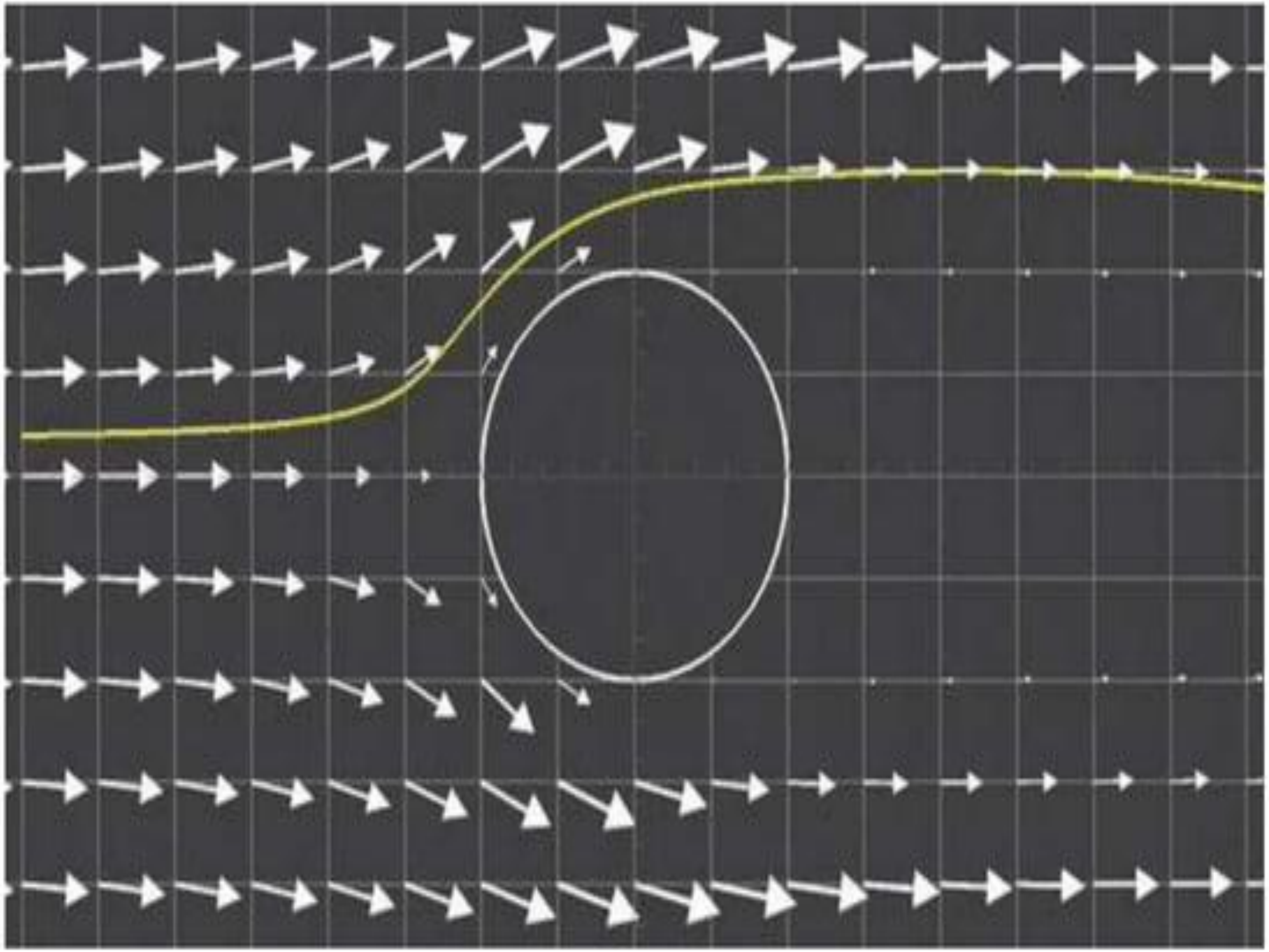
$u=dx/dt$;$v=dy/dt$; $w=dz/dt$

$a_x=du/dt=d^2x/dt^2$;$a_y=d^2y/dt^2$; $a_z=d^2z/dt^2$

Resultant velocity=?

Resultant acceleration=?


Eulerian Frame
Lagrangian Frame



Eulerian method

- In this method, a point or section is chosen in the flow field
- Its velocity, acceleration, density etc. are observed at that point
- It is a commonly used method because of mathematical simplicity Let u, v, and w be the components of resultant velocity in x, y and z axis respectively. The velocity components vary along space and time.

$$u=f1(x,y,z,t);$$

$$v=f2(x,y,z,t);$$

$$w=f3(x,y,z,t)$$

Then, resultant velocity is given by

Resultant velocity=?

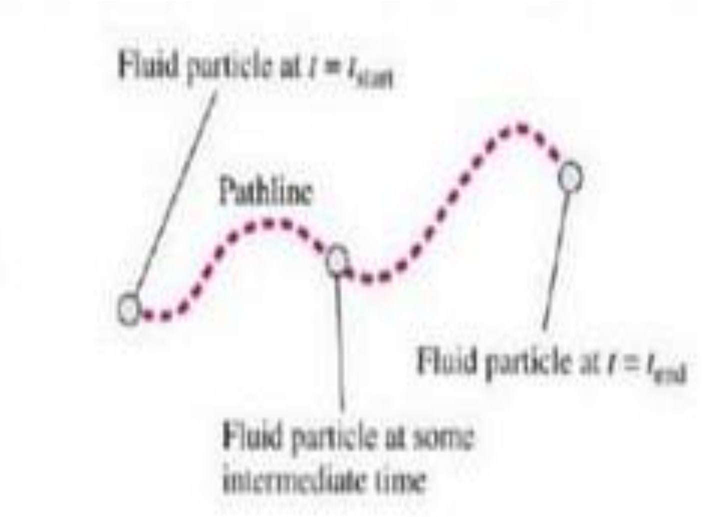
Flow Visualization is the Visual Examination of Flow-Field Features.

It is important for both physical experiments and numerical solutions.

Description of Flow Patterns:

1. Stream Lines
2. Path Lines
3. Streak Lines
4. Stream Tube

- **Path line:** It is trace made by single particle over a period of time.
- **Streamline** show the mean direction of a number of particles at the same instance of time.

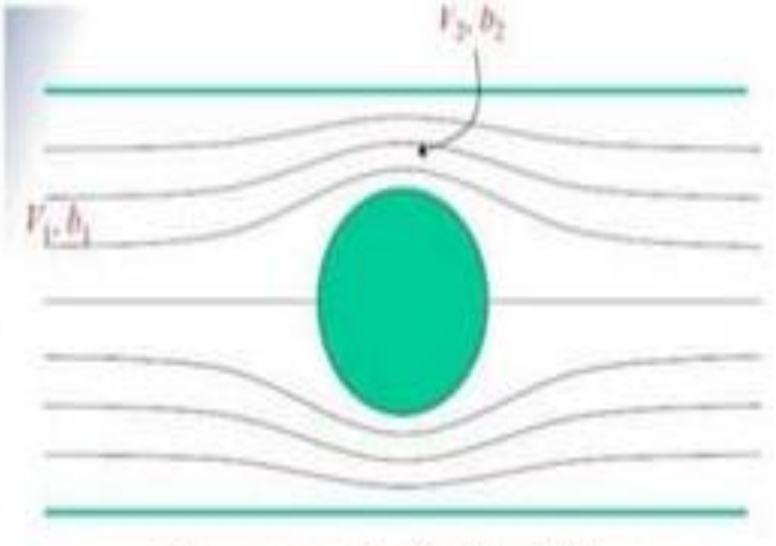


➢ Character of Streamline

1.Streamlines can not cross each other. (otherwise, the cross point will have two tangential lines.)

2. Streamline can't be a folding line, but a smooth curve.

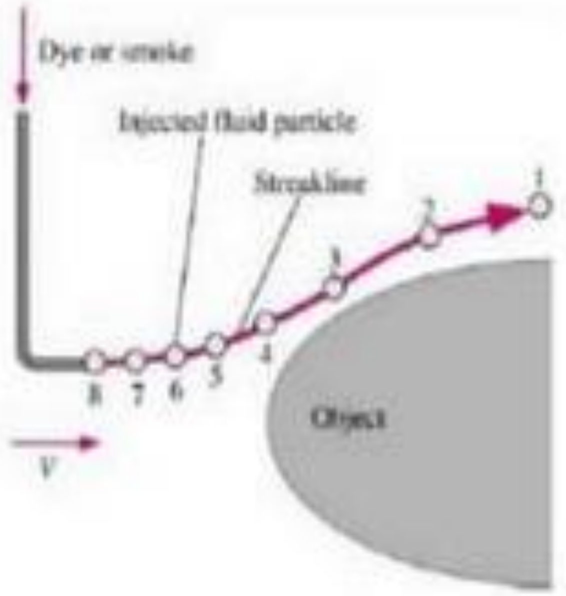
3. Streamline cluster density reflects the magnitude of velocity. (Dense streamlines mean large velocity: while sparse streamlines mean small velocity.)



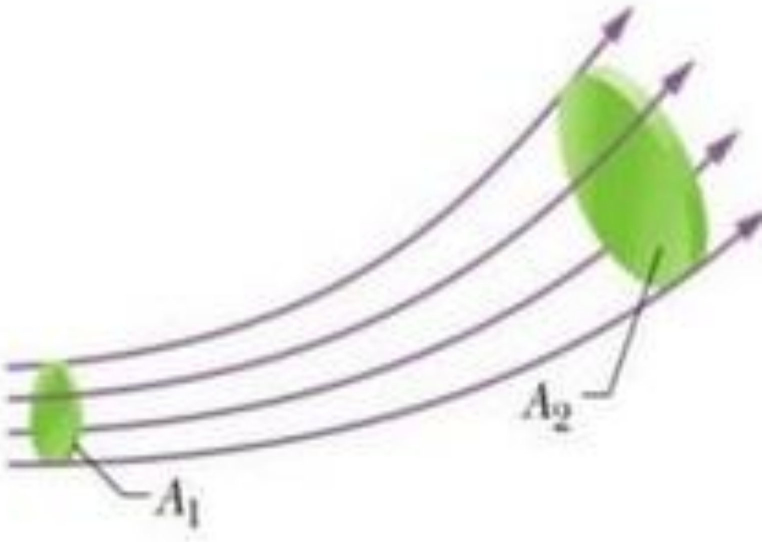
Flow around cylindrical object

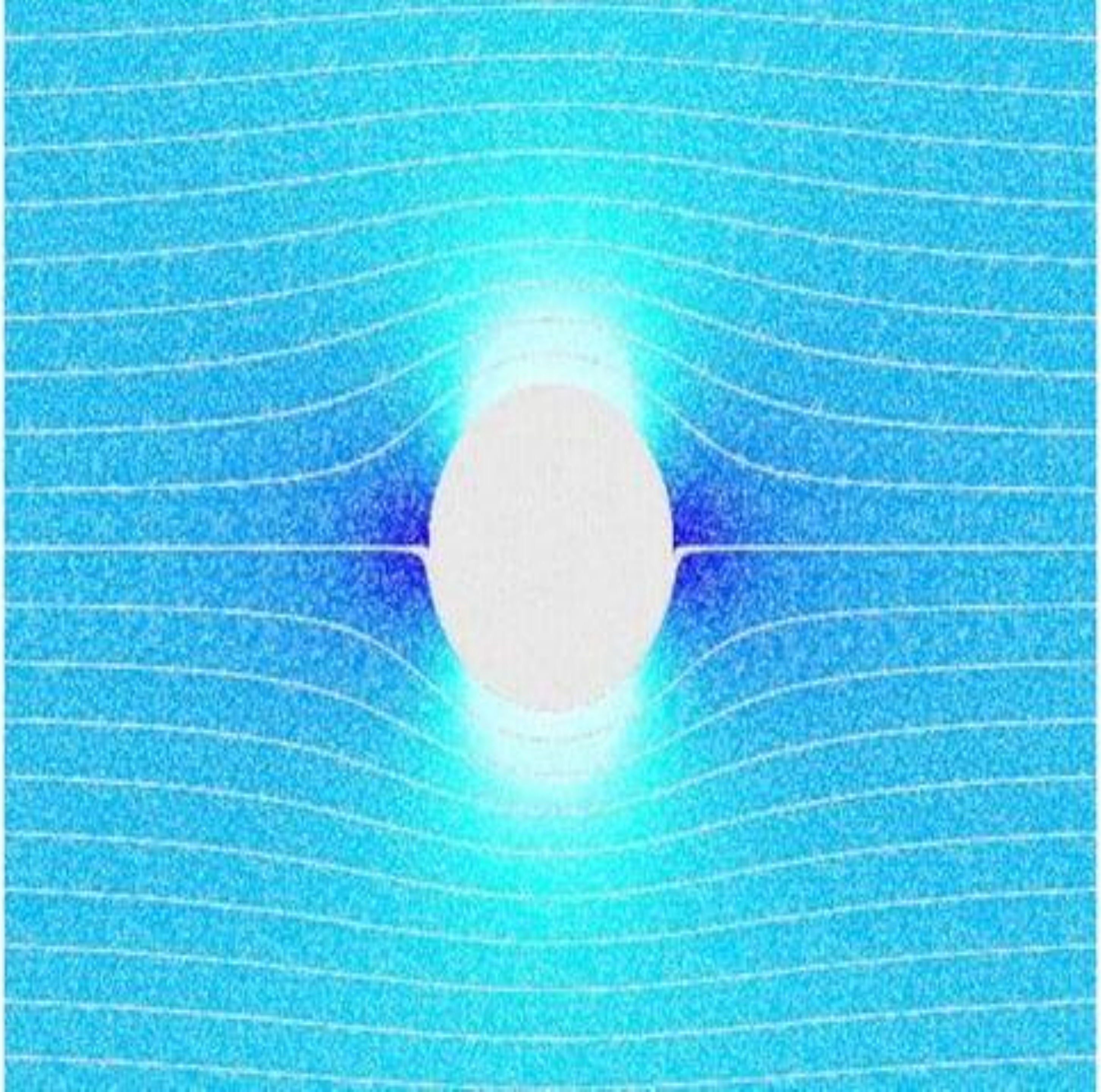
› A **Streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.

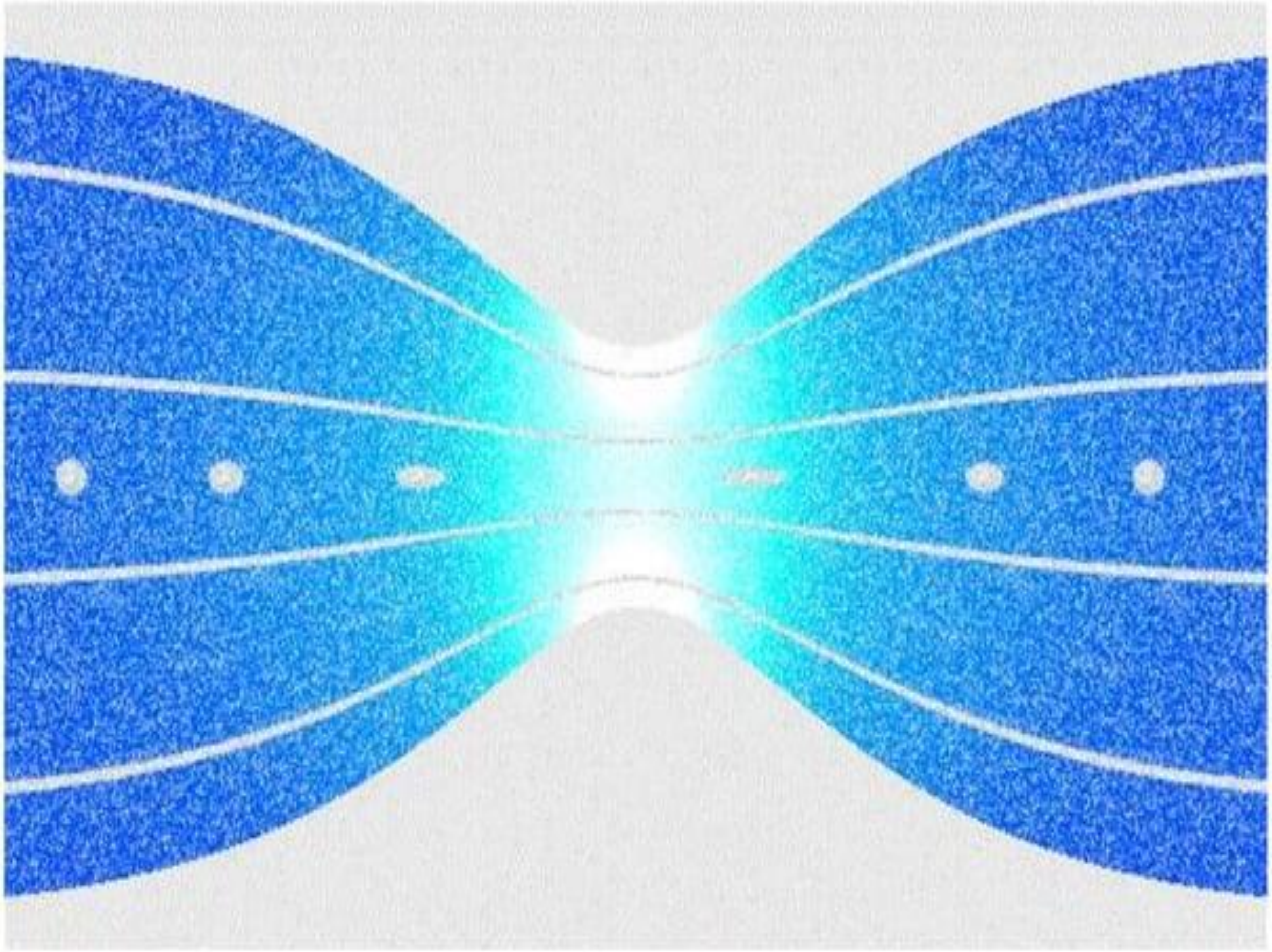
It is an instantaneous picture of the position of all particles in flow that have passed through a given point.



- Streamtube is an imaginary tube whose boundary consists of streamlines.
- The volume flow rate must be the same for all cross sections of the stream tube.







Types of Fluid Flow

1. Steady and Unsteady Flow
2. Uniform and Non-Uniform Flow
3. Laminar and Turbulent Flow
4. Rotational and Irrotational Flow
5. Compressible and Incompressible Flow
6. Ideal and Real Flow
7. One, Two and Three Dimensional Flow

Steady flow is that type of flow in which fluid parameters (velocity, pressure, density etc.) at any point in the flow field do not change with time.

This means that the fluid particles passing through a fixed point have the same flow parameters like velocity, pressure, surface tension etc.

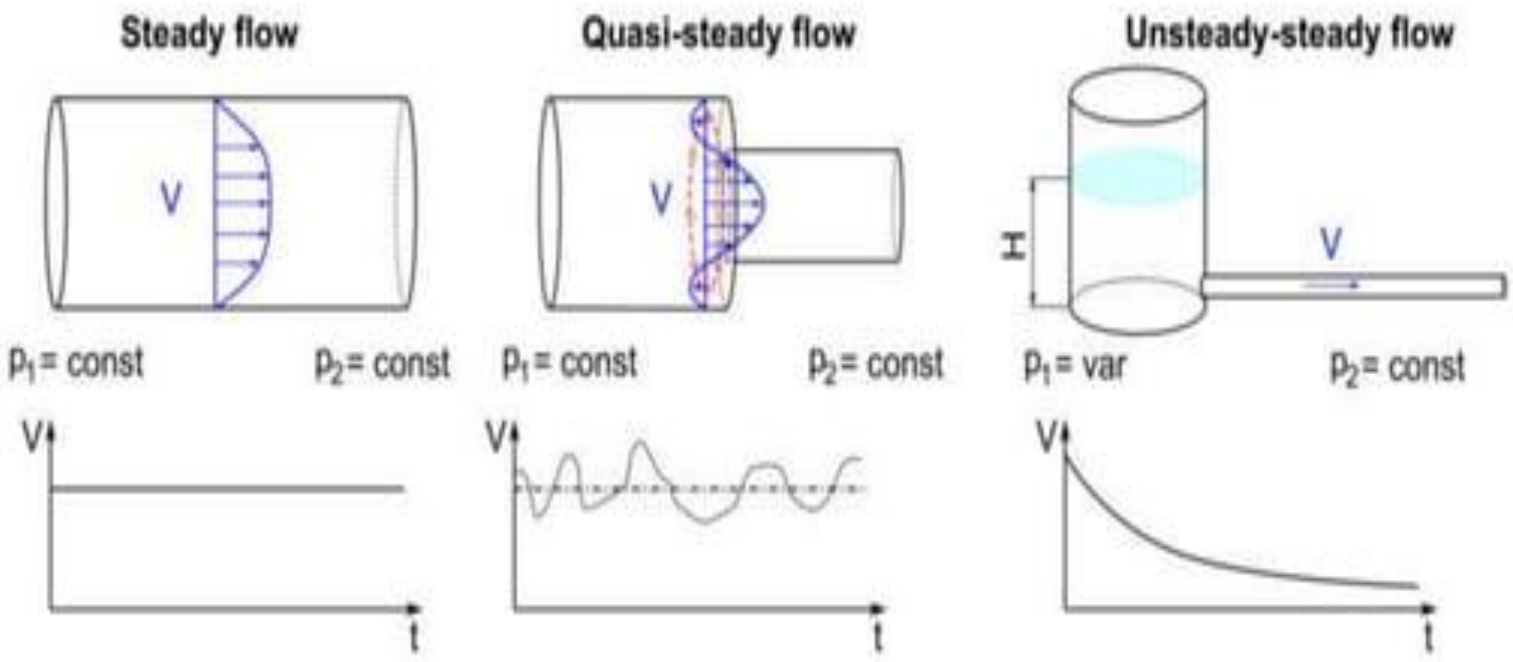
The parameters may be different at the different cross-section of the flow passage.

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

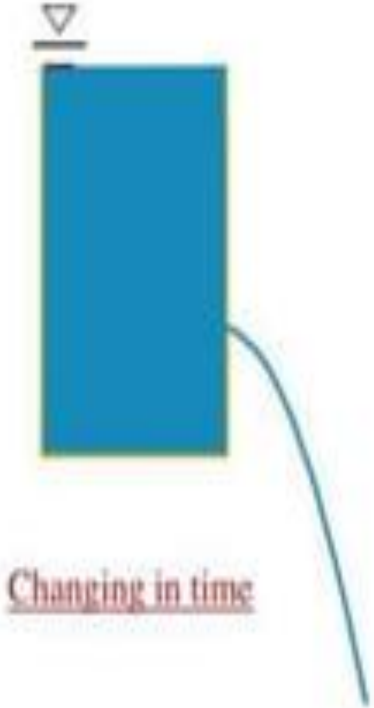
where (x_0, y_0, z_0) is a fixed point in fluid field.

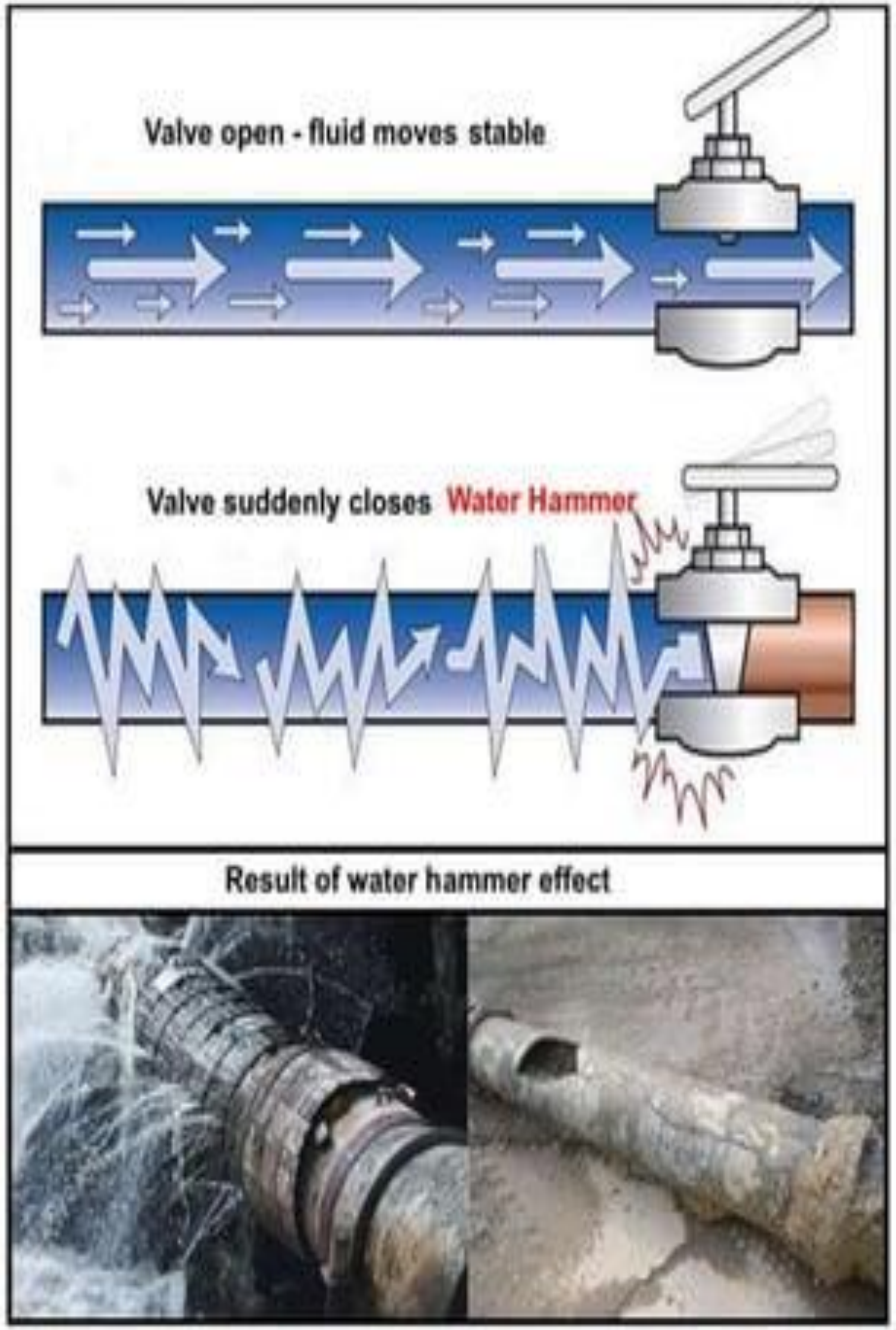
Unsteady flow is that type of flow in which fluid parameters (velocity, pressure, density etc.) at a point changes with time.

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} \neq 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0 \text{ etc.}$$

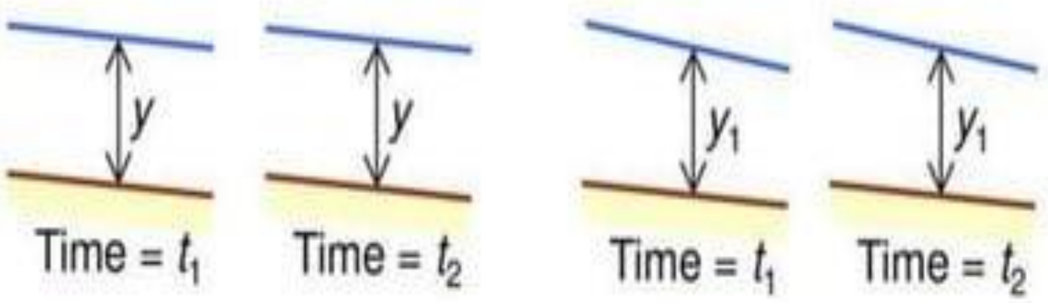


A flow is said to be **quasi steady** if temporal variations at a spatial location are much smaller (they would be zero if the flow was steady) compared to spatial variations for any quantity.

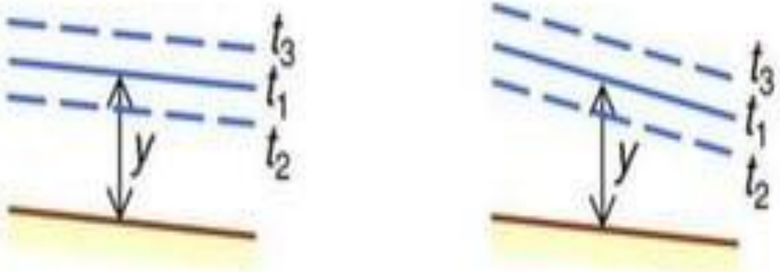


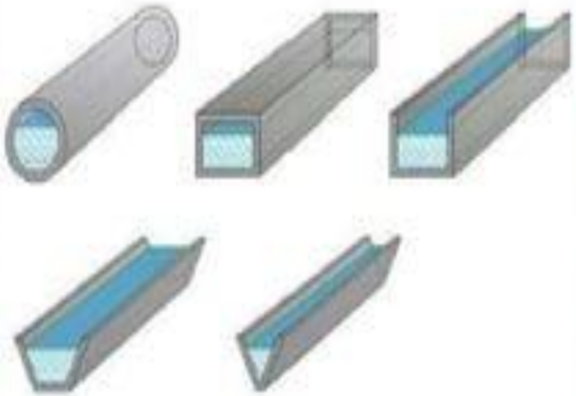

Valve open - fluid moves stable
Valve suddenly closes
Water Hammer
Result of water hammer effect

Steady flow



Unsteady flow





Uniform and Non-uniform Flows.

Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (i.e., length of direction of the flow). Mathematically, for uniform flow

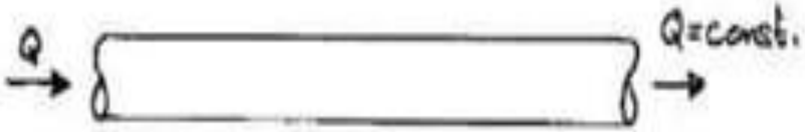
$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} = 0$$

where ∂V = Change of velocity

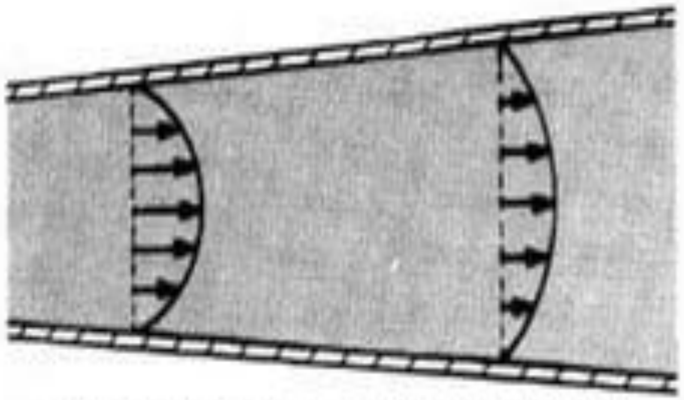
∂s = Length of flow in the direction S.

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

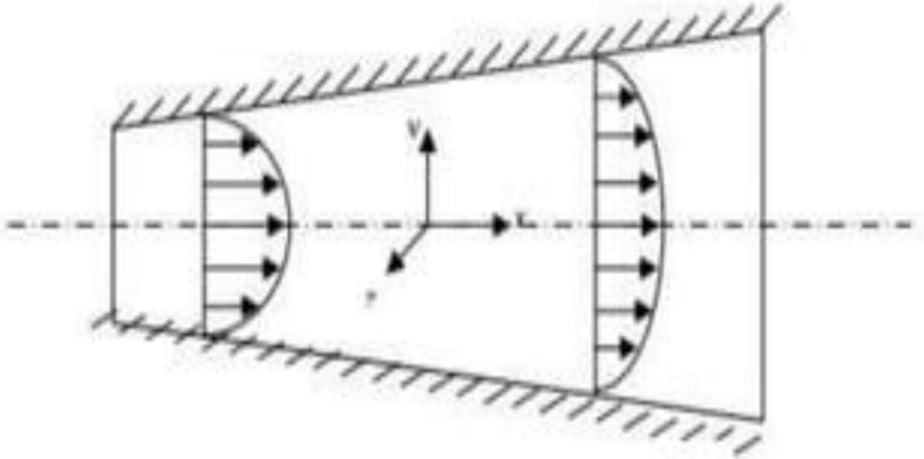
$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} \neq 0.$$



Steady uniform flow



Steady non-uniform flow



Three dimensional flow

Laminar flow is also called streamline or viscous flow. This type of flow occurs in smooth pipes having the low velocity of flow. It also occurs in liquids having high viscosity.

Turbulent flow is defined as that type of flow in which each fluid particle does not have a definite path and the paths of individual particles cross each other.

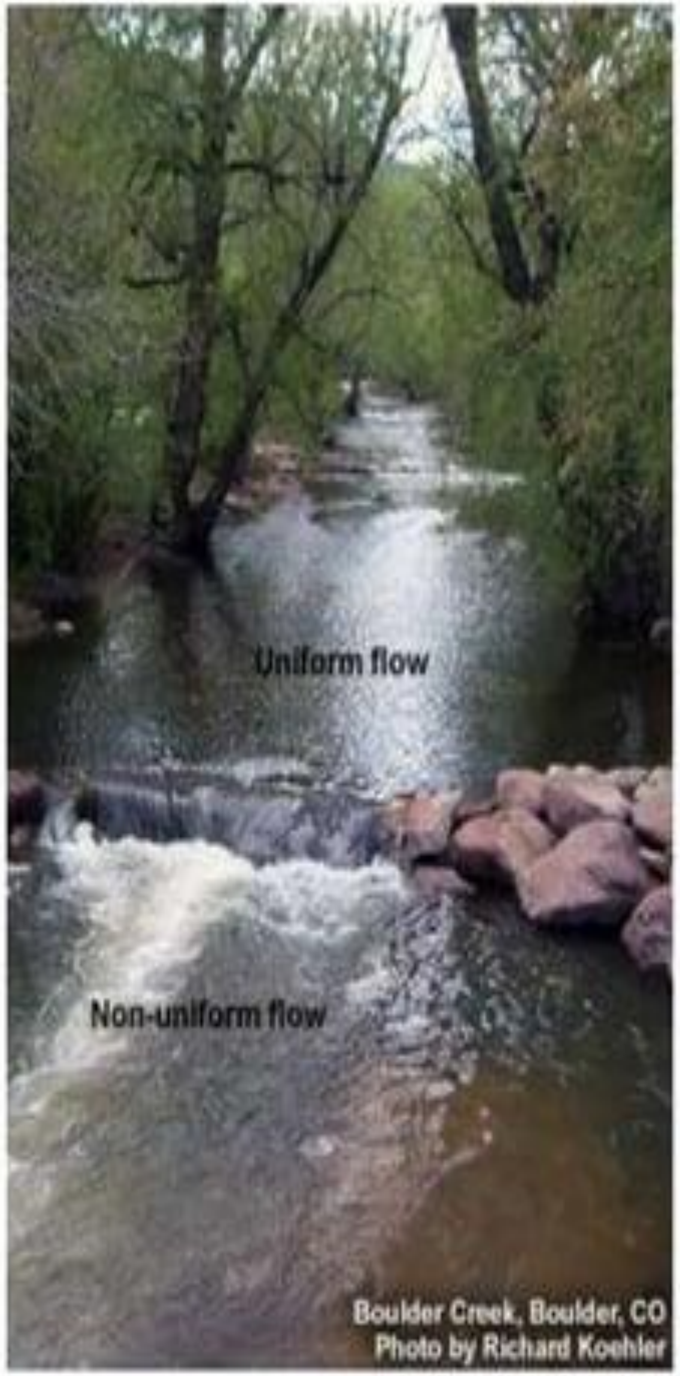
In other words, it is the flow in which fluid particles move in a zigzag path

When a fluid is flowing in a pipe, the type of flow is determined by a non-dimensional number, called Reynold's number.

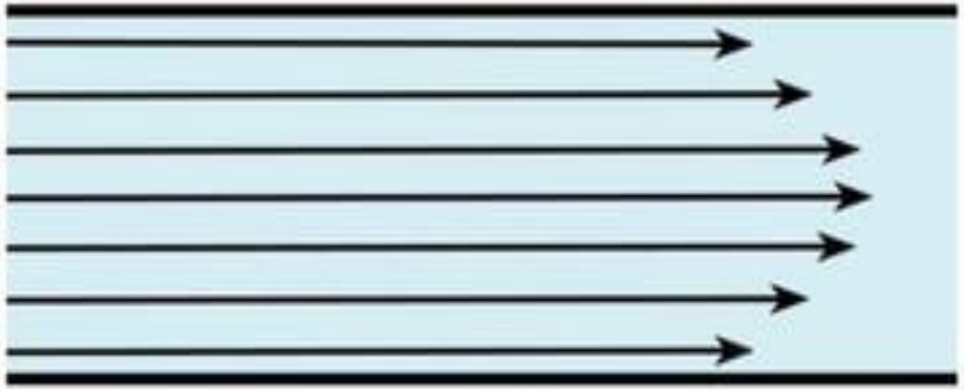
For laminar flow, Reynold number < 2000

For turbulent flow Reynold number > 4000

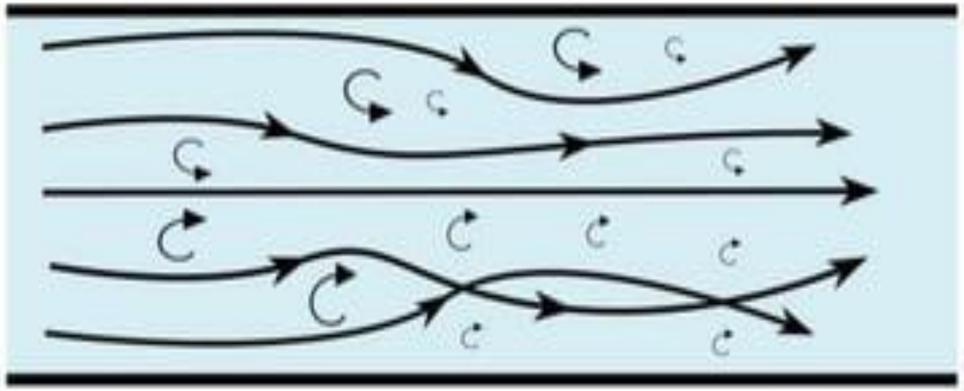

Uniform and Non-Uniform Flow
Uniform flow
Non-uniform flow
Flow direction
©The COMET Program

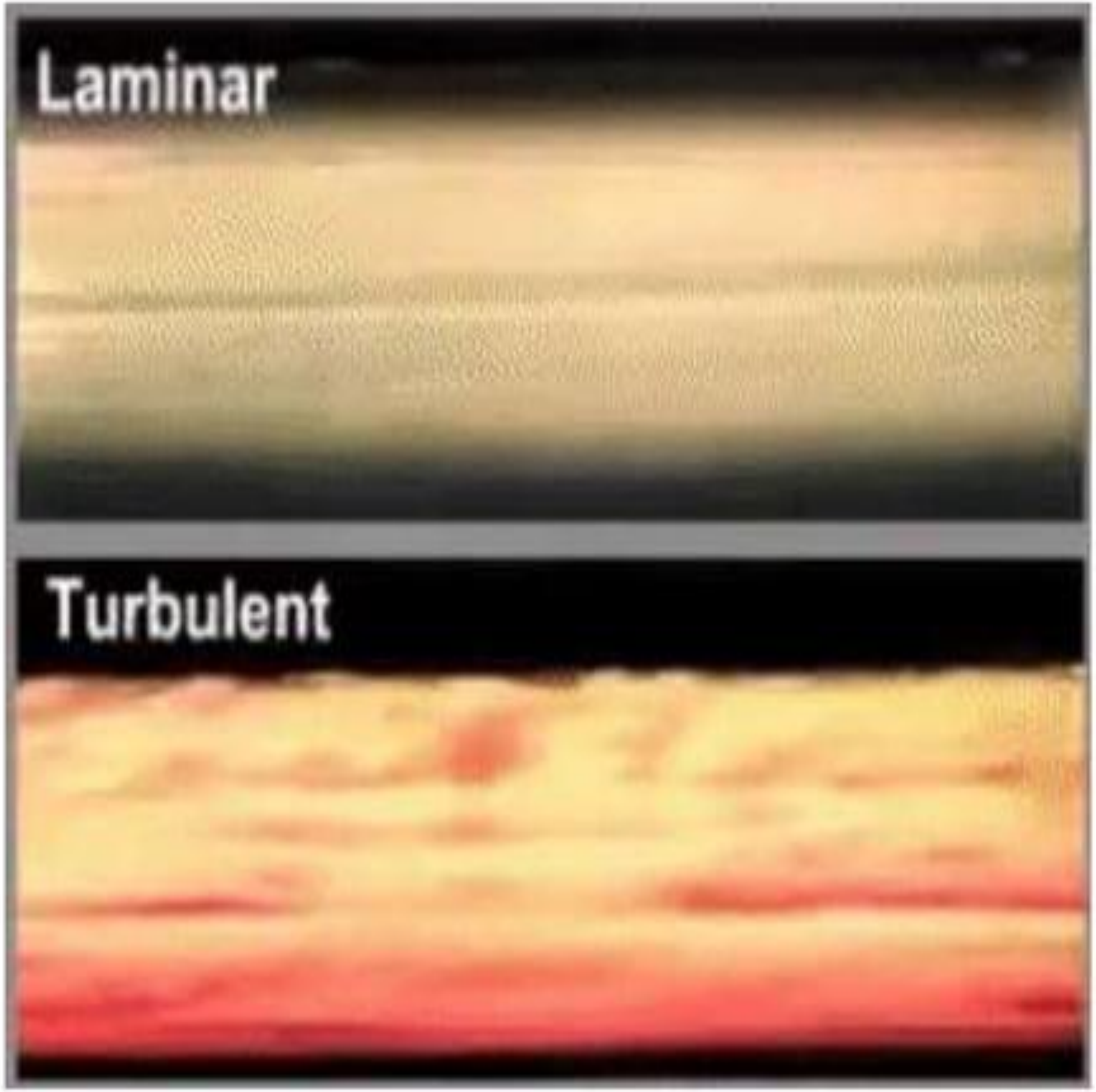

Uniform flow
Non-uniform flow
Boulder Creek, Boulder, CO
Photo by Richard Koehler

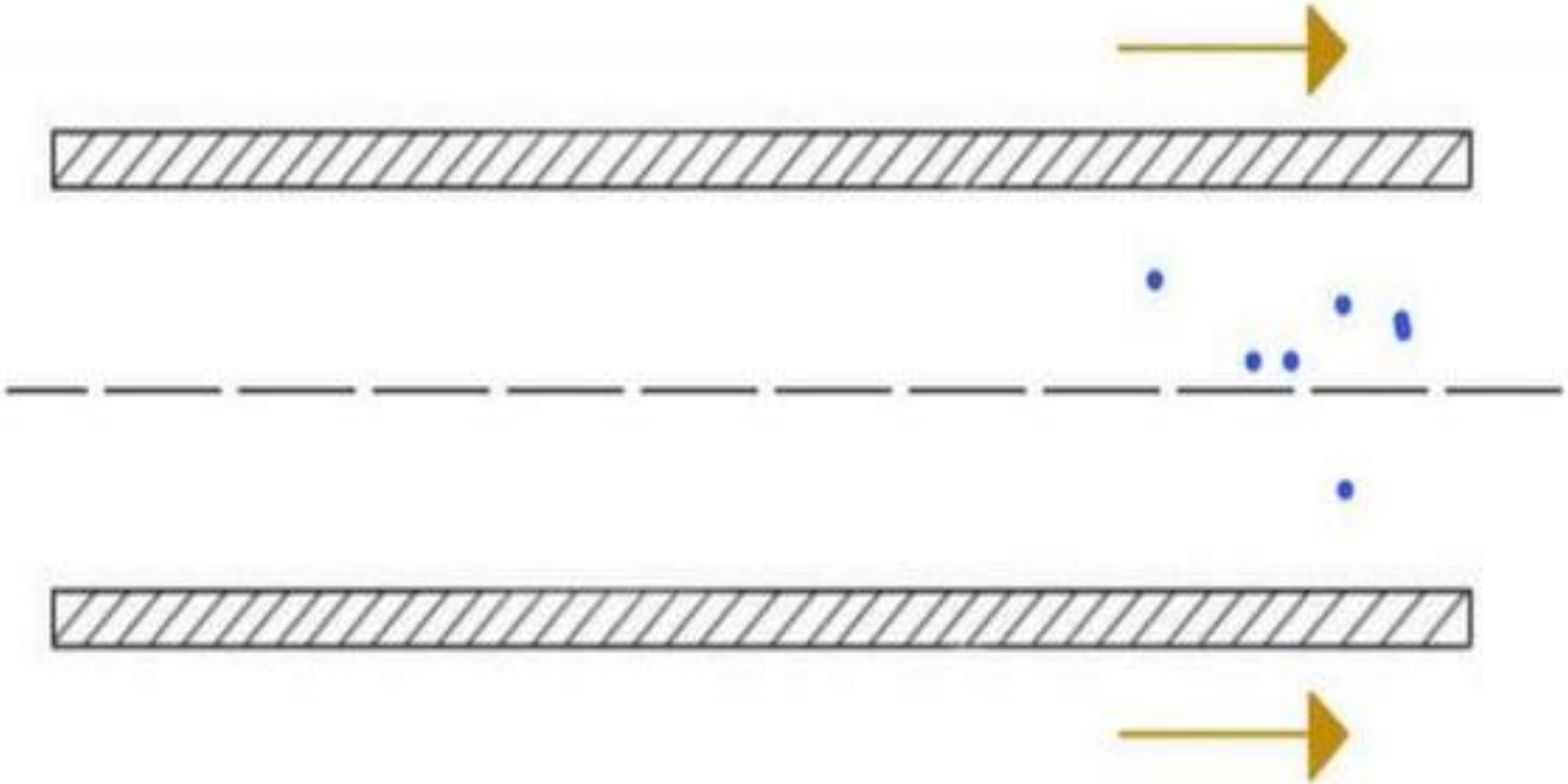
laminar flow



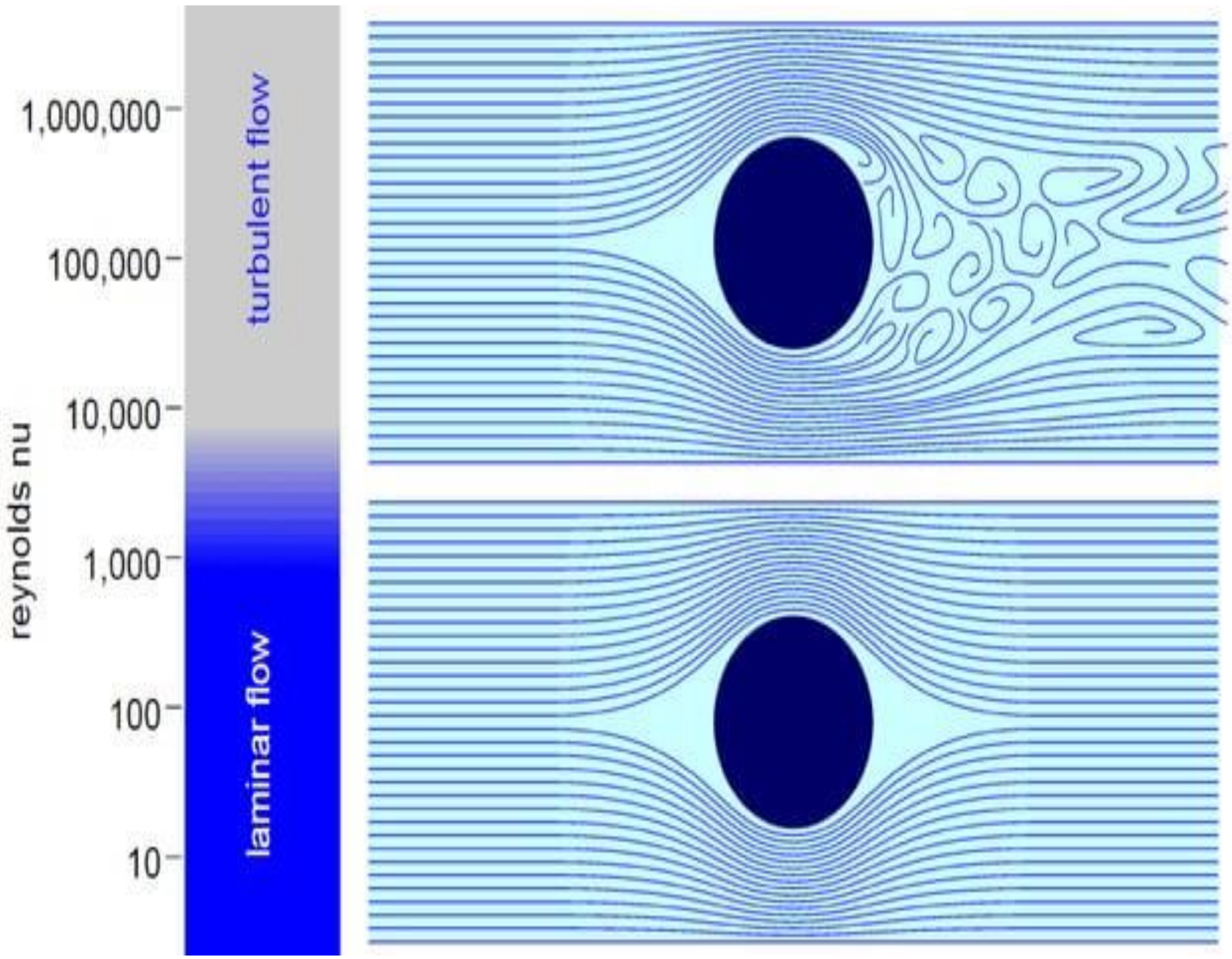
turbulent flow




Laminar
Turbulent



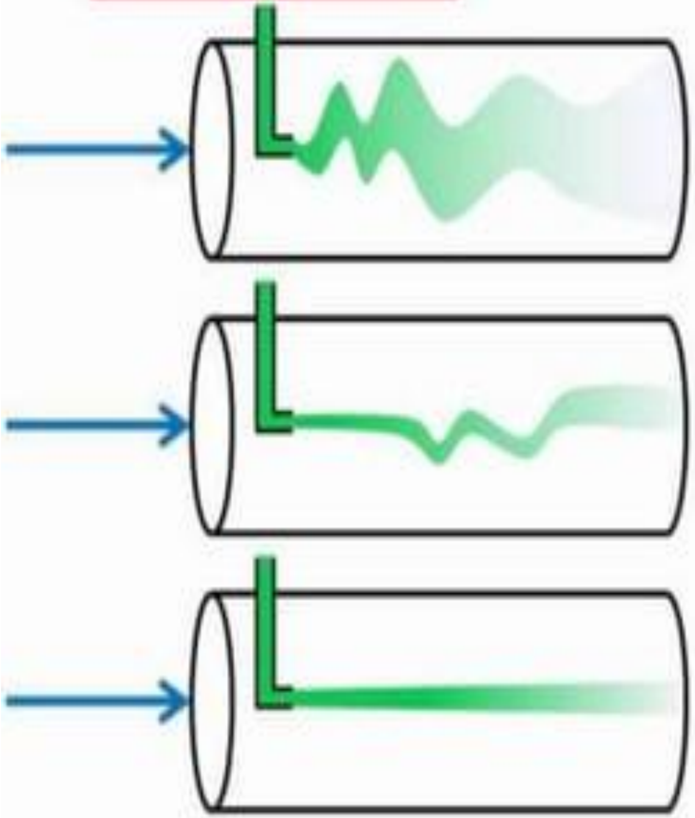
turbulent flow



reynolds nu
1,000,000
100,000
10,000
1,000
100
10
turbulent flow
laminar flow

The Reynolds number correlates well with flow characteristics.

$$\mathrm{Re} = \frac{\rho V_{avg} D}{\mu}$$



$\mathrm{Re} > 4000$

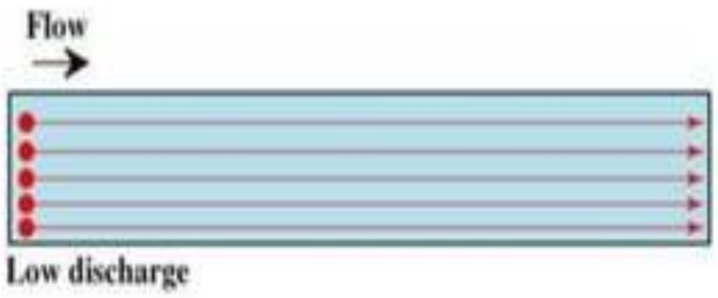
turbulent (unpredictable, rapid mixing)

$2300 < \mathrm{Re} < 4000$

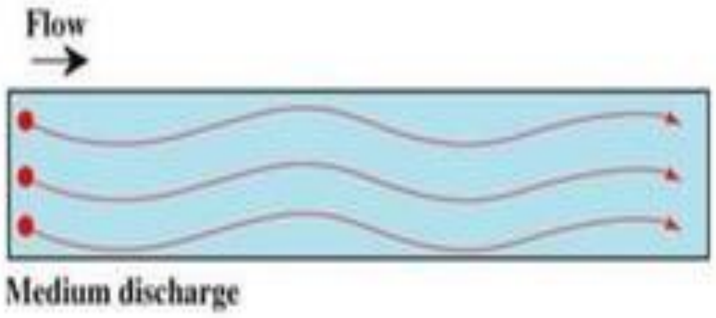
transitional (turbulent outbursts)

$\mathrm{Re} < 2300$

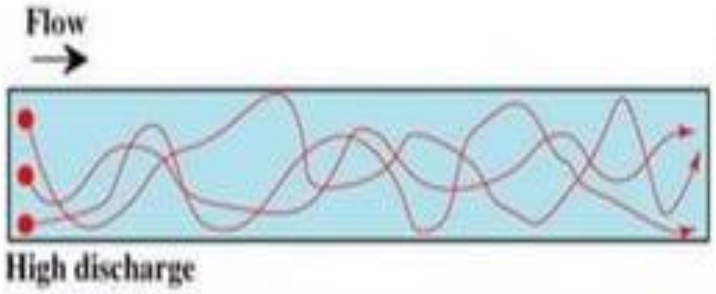
laminar (predictable, slow mixing)



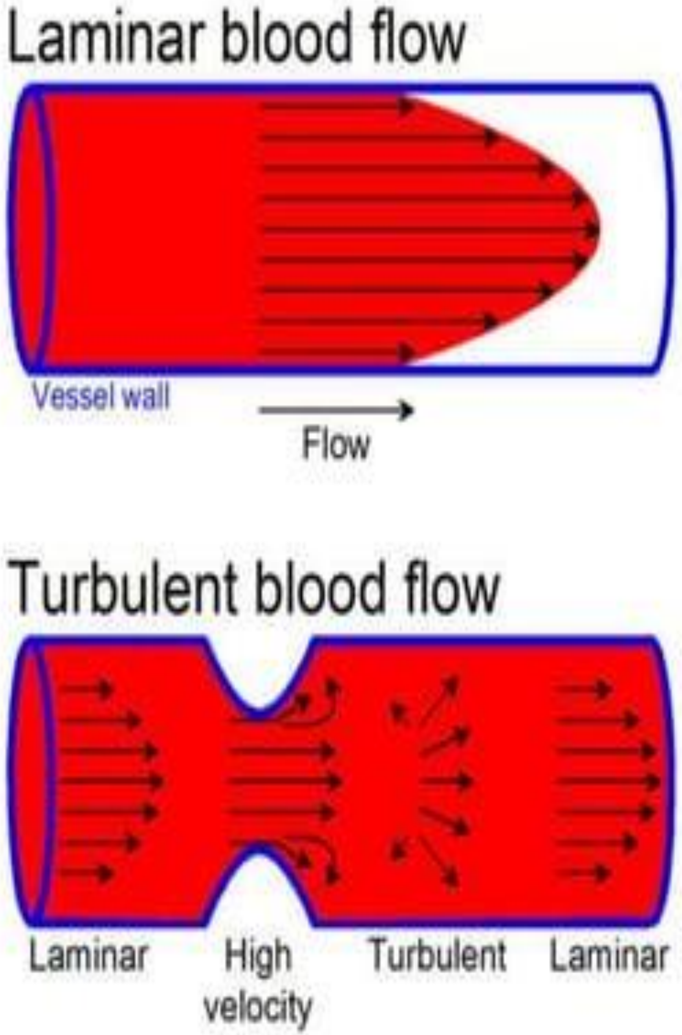
Laminar Flow: every fluid molecule followed a straight path that was parallel to the boundaries of the tube.



Transitional Flow: every fluid molecule followed wavy but parallel path that was not parallel to the boundaries of the tube.



Turbulent Flow: **E**very fluid molecule followed very complex path that led to a mixing of the dye.


Laminar blood flow
Vessel wall
Flow
Turbulent blood flow
Laminar
High velocity
Turbulent
Laminar

compressible flow: The flow in which the density of fluid changes, due to pressure and temperature variations, from point to point during the flow is called compressible flow.

In other words, it is the flow in which the density of a fluid is not constant during the flow.

Mathematically, for compressible flow,

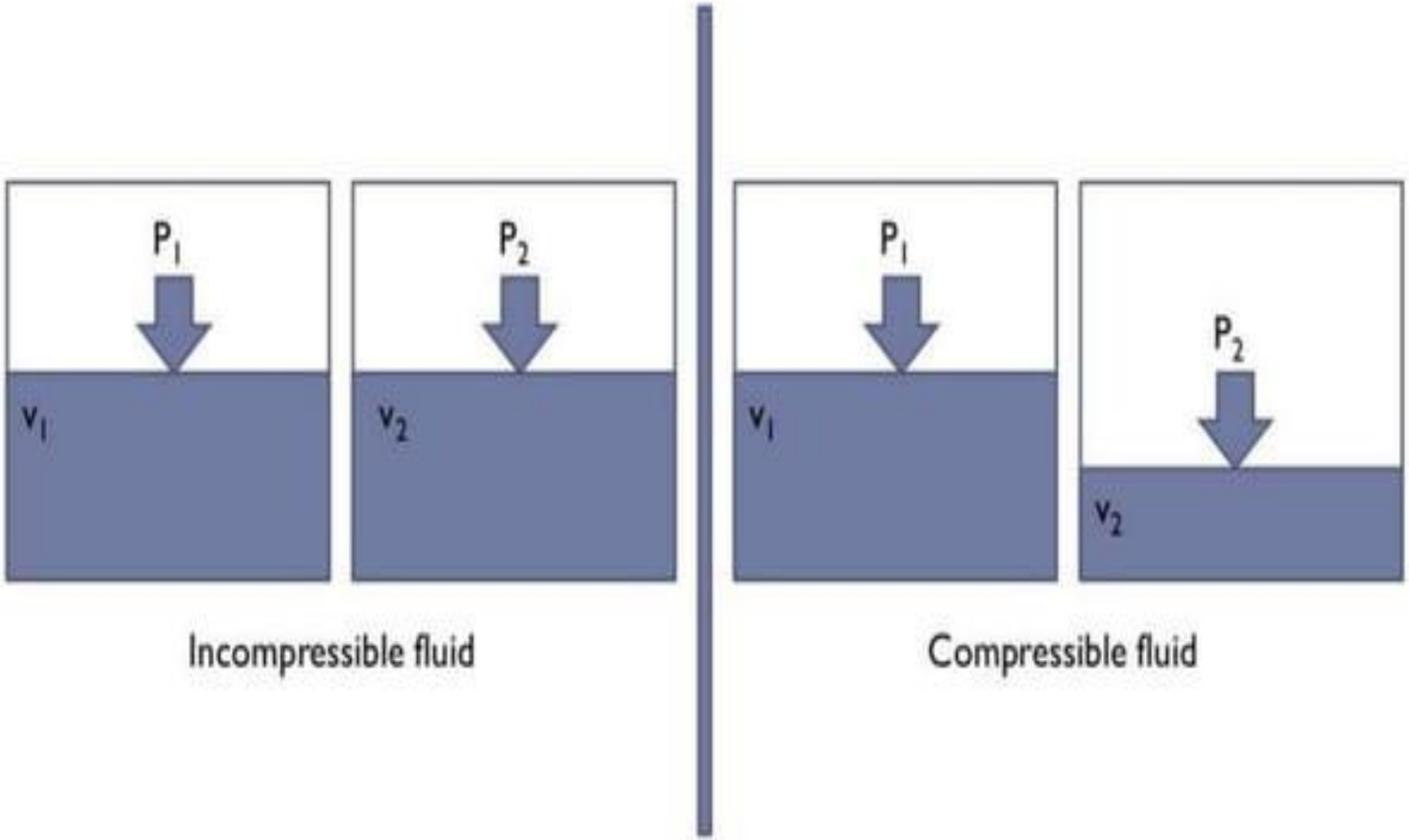
$$\rho \neq \text{constant}$$

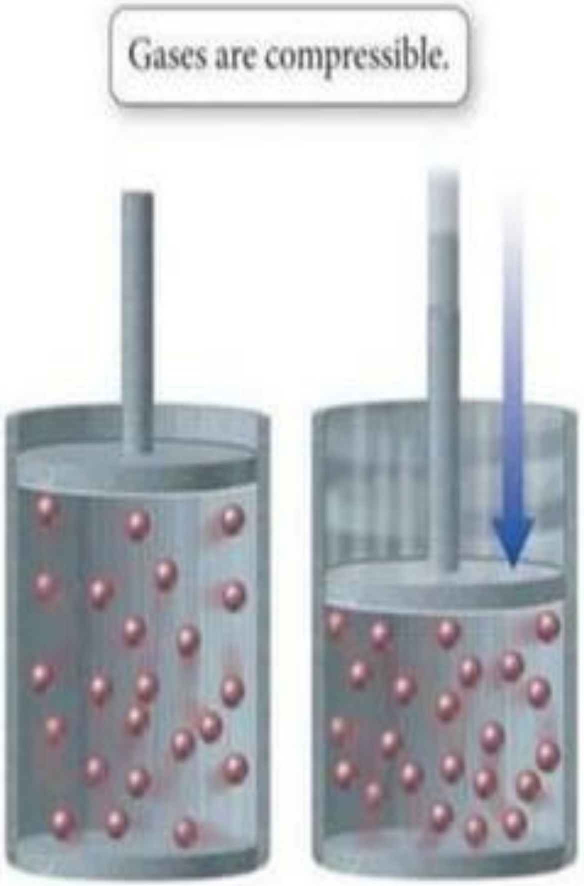
incompressible flow : The flow in which the density of fluid does not change during the flow is called incompressible flow. In other words, it is the flow in which the density of a fluid is constant during the flow.

Mathematically, for incompressible flow,

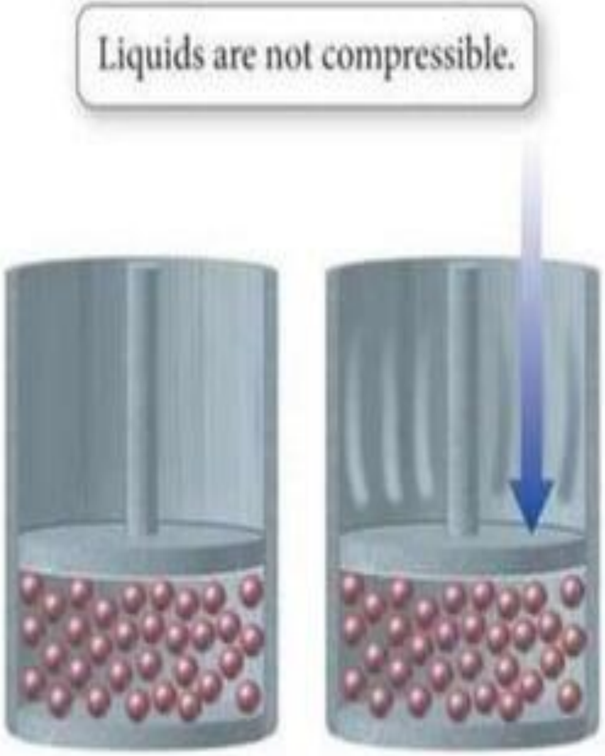
$$p = \text{constant}$$

Liquids are generally incompressible which means that pressure and temperature changes have a very little effect on their volume. Gases are compressible fluids.


P1
P2
P1
P2
v1
v2
v1
v2
Incompressible fluid
Compressible fluid



Gas

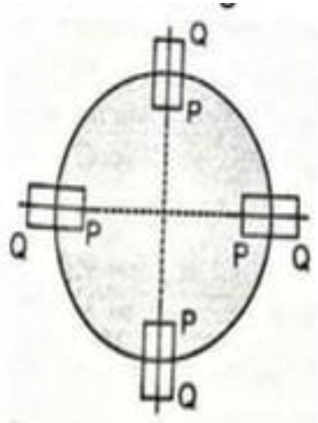


Liquid

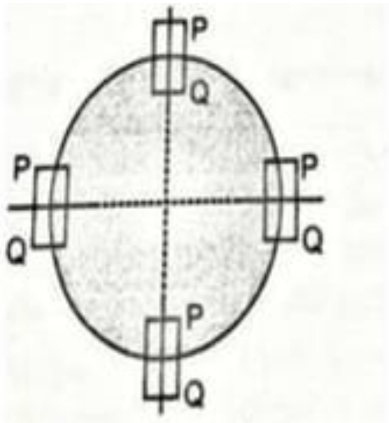
Compressible Flow

Rotational flow is that type of flow in which fluid particles also rotate about their own axes while flowing along a streamline.

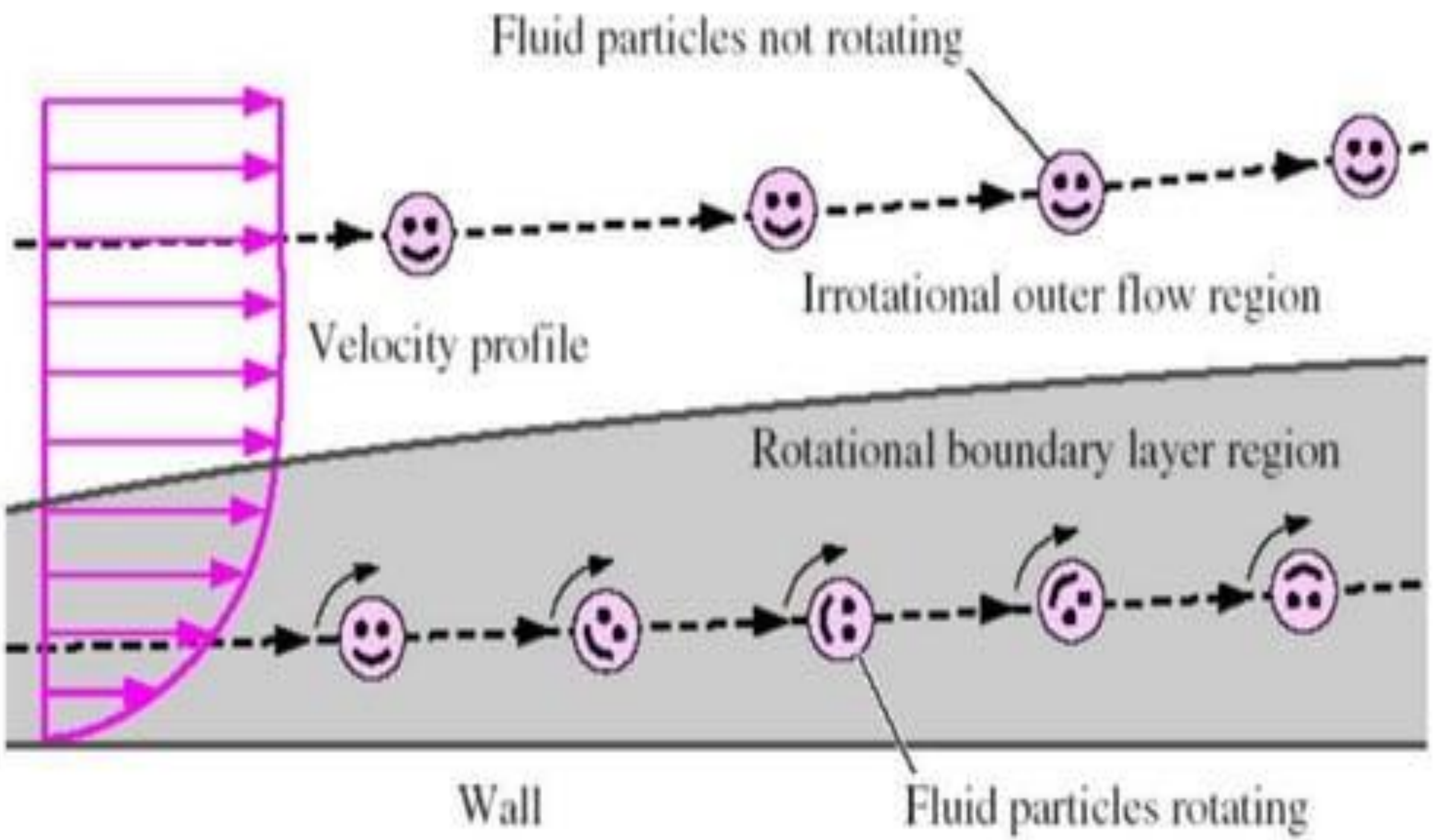
Irrotational flow is that type of flow in which fluid particles do not rotate about their own axes while flowing.

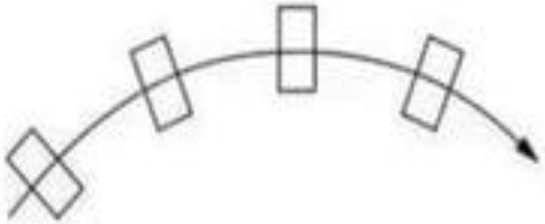


Rotational flow

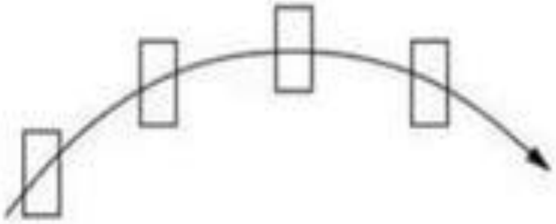


Irrotational flow

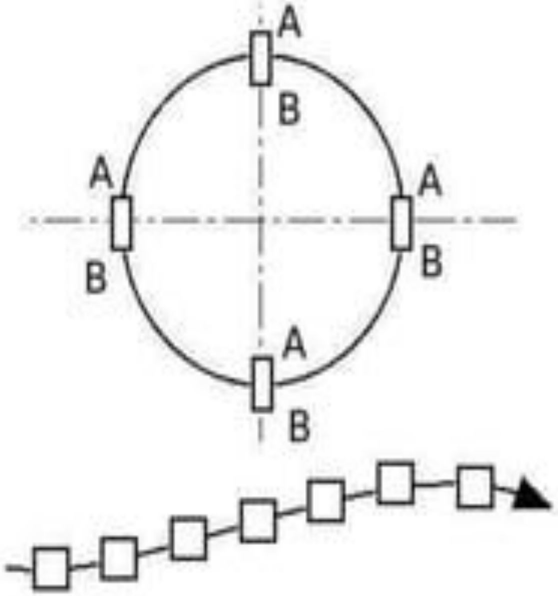

Fluid particles not rotating
Irrotational outer flow region
Velocity profile
Rotational boundary layer region
Wall
Fluid particles rotating



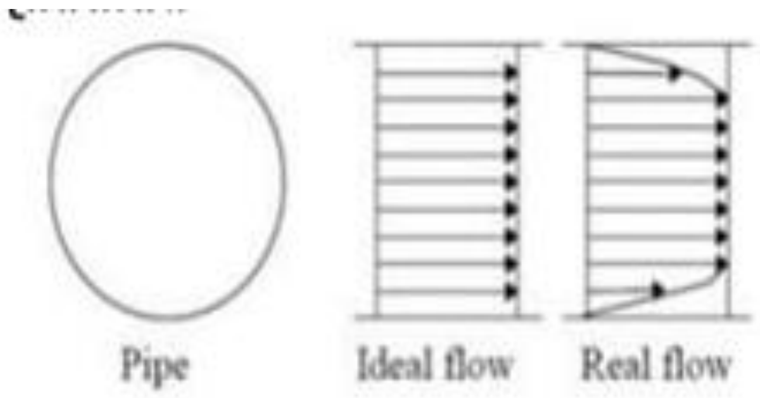
Rotational flows



Irrotational flows

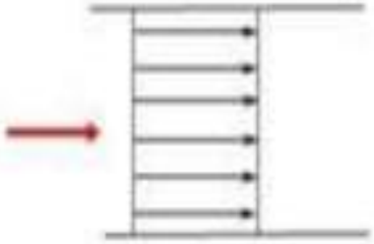


- An **ideal flow** is the flow of a non-viscous fluid. In the ideal flow, **no shear stress** exists between two adjacent layers or between the fluid layer and boundary, only normal stresses can exist in ideal flows.
- The **flow of real (viscous) fluids** is called real flow. In real flow, shear stress exists between to adjacent fluid layers. These stresses oppose the sliding of one layer over another



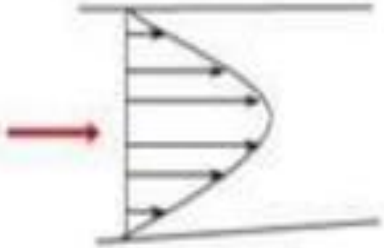
Velocity distribution of pipe flow

Real fluid flows implies friction effects. Ideal fluid flow is hypothetical; it assumes no friction.



Ideal

Friction = 0
Ideal Flow (μ =0)
Energy loss =0



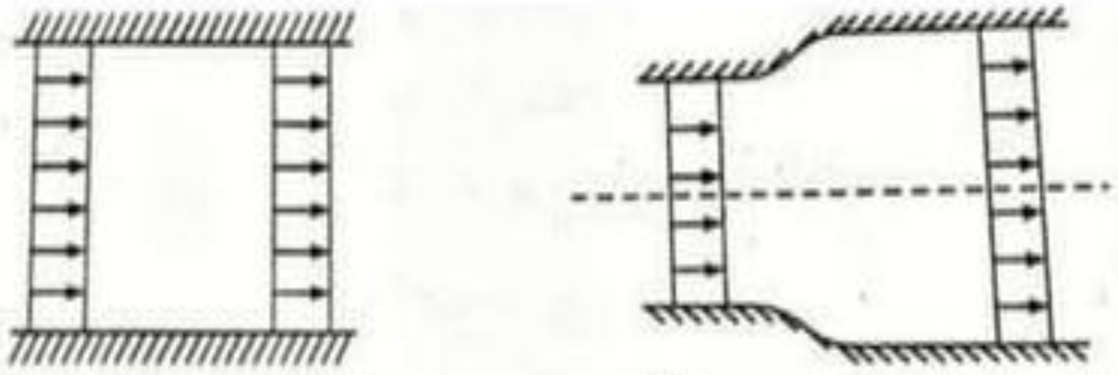
Real

Friction ≠ o
Real Flow (μ ≠0)
Energy loss ≠ 0

One dimensional flow is the flow in which parameters (velocity, pressure, density, viscosity and temperature) vary only in one direction and the flow is a function of only one co-ordinate Axis and time. The flow field is represented by streamlines which are straight and parallel

mathematically, for one-dimensional flow

$$u = f(x),\ v = 0 \text{ and } w = 0$$

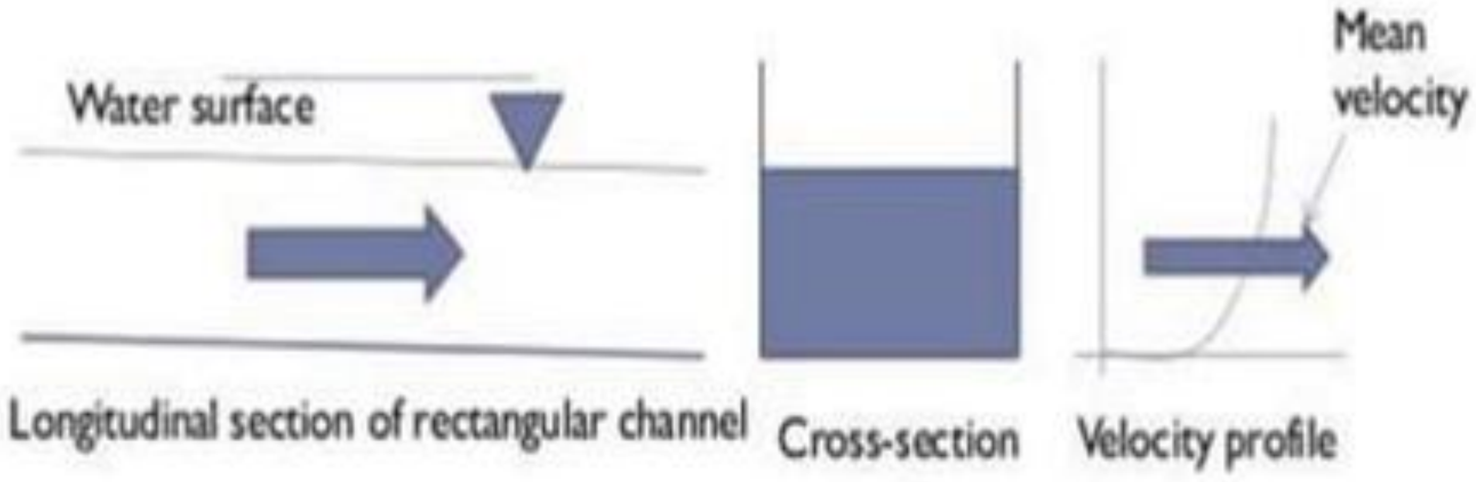


(a) Flow between parallel plates (b) incompressible flow in a duct

Illustration of one dimensional flow

Although in general **all fluids flow three-dimensionally**, with pressures and velocities and other flow properties varying in all directions, in many cases the greatest changes only occur in two directions or even only in one. In these cases changes in the other direction can be effectively ignored making analysis much more simple.

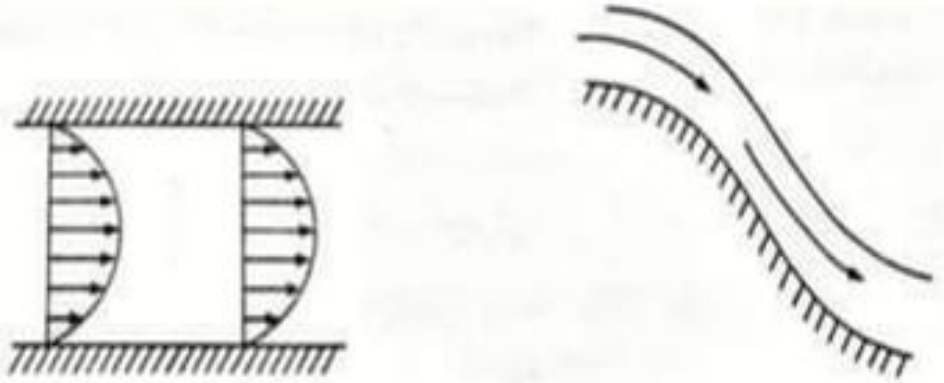
Flow is one dimensional if the flow parameters (such as velocity, pressure, depth etc.) at a given instant in time only vary in the direction of flow and not across the cross-section



Two-dimensional flow is the flow in which fluid parameters vary along two directions and the flow is the function of two rectangular space coordinates (x and y- axis) and time.

The flow field is represented by streamlines which are curves. mathematically for two-dimensional flow

$$u = f_1(x, y),\ v = f_2(x, y) \text{ and } w = 0.$$



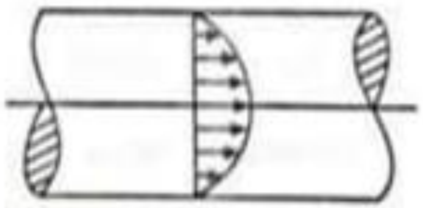
(a) Viscous flow between parallel plates

(b) Flow over a spillway

Illustration of two dimensional flow

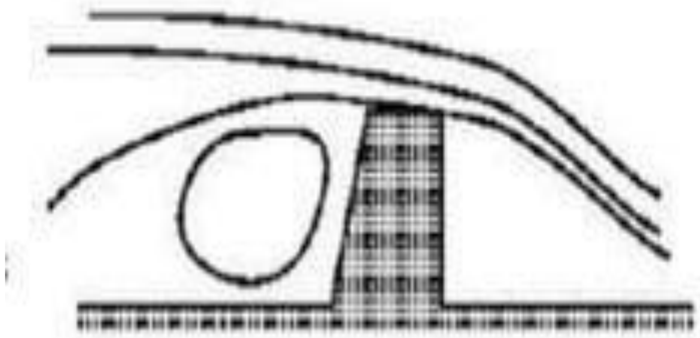
Three-dimensional flow is the flow in which flow parameters change in all the three directions and the flow is the functions of three mutually perpendicular co-ordinate Axis (x, y, z-axis) and time. The streamlines are space curves

$$u = f_1(x, y, z),\ v = f_2(x, y, z) \text{ and } w = f_3(x, y, z).$$



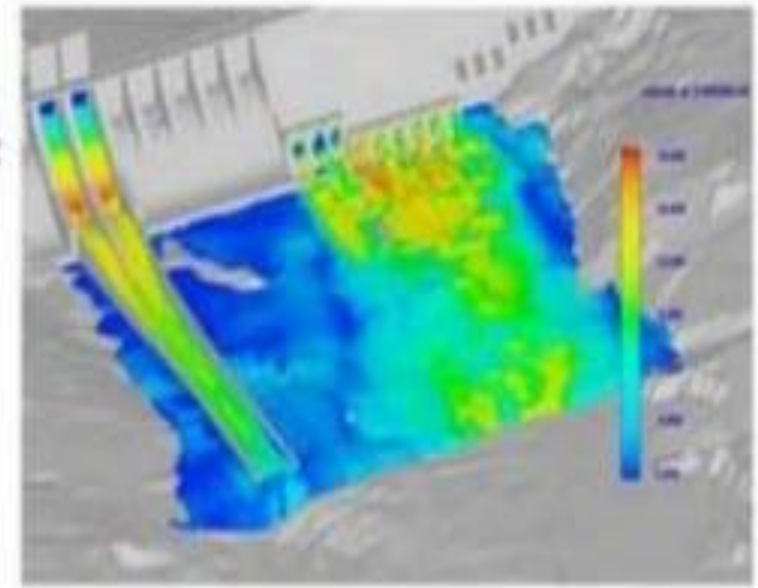
Viscous flow in a duct (three dimensional flow)

- Flow is two-dimensional if it can be assumed that the flow parameters vary in the direction of flow and in one direction at right angles to this direction



Two-dimensional flow over a weir

Flow is three-dimensional if the flow parameters vary in all three directions of flow



Three-dimensional flow in stilling basin

Rate of Flow or Flow Rate or Discharge:

Quantity of fluid passing through any section in a unit time.

Unit: m^3/sec

Types:

1. Volume Flow Rate = Volume of fluid / Time

   = Q or AV

2. Mass Flow Rate = Mass of fluid / Time = ρ x Volume / Time =

   ρ Q or PAV

3. Weigh Flow Rate = Weight of fluid / Time = (pg x Volume) / Time

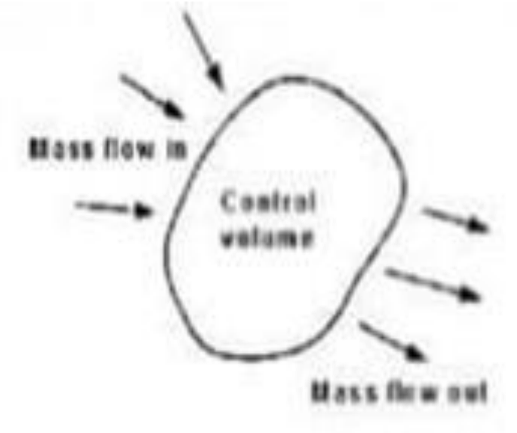
   = pgQ or pgAV   Where, Q = Area x Velocity

Continuity

- Matter cannot be created or destroyed

-(it is simply changed in to a different form of matter).

- This principle is know as the conservation of mass and we use it in the analysis of flowing fluids.
- The principle is applied to fixed volumes, known as control volumes shown in figure:
- For any control volume the principle of conservation of mass says



An arbitrarily shaped control volume.

Mass entering per unit time - Mass leaving per unit time = Increase of mass in the control volume per unit time

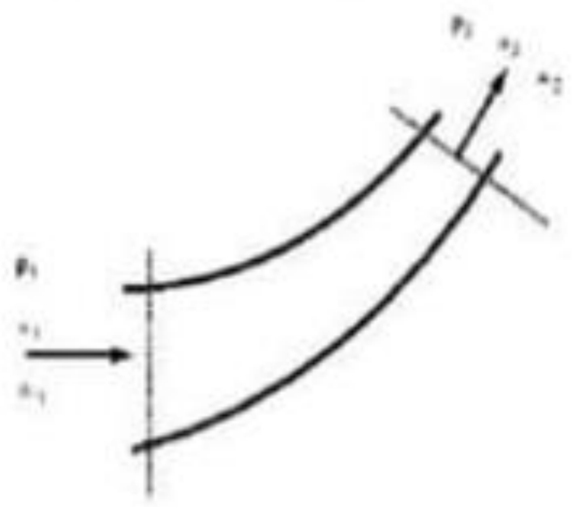
Continuity Equation

- For steady flow there is no increase in the mass within the control volume, so Mass entering per unit time = Mass leaving per unit time

- **Derivation:**

Lets consider a stream tube.

$\rho_1 . V_1$ and A_1 are mass density. velocity and cross-sectional area at section 1. Similarly, $\rho_2 . V_2$ and A_2 are mass density, velocity and cross-sectional area at section 2.



A stream tube

$$M_1 = \rho_1 A_1 V_1$$
$$M_2 = \rho_2 A_2 V_2$$

- According to mass conservation

$$M_1 - M_2 = \frac{d(M_{CV})}{dt}$$
$$\rho_1 A_1 V_1 - \rho_2 A_2 V_2 = \frac{d(M_{CV})}{dt}$$

- For steady flow condition $d(M_{cv})/dt = 0$

$$\rho_1 A_1 V_1 - \rho_2 A_2 V_2 = 0 \Rightarrow \rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

$$M = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

- Hence, for stead flow condition, mass flow rate at section 1= mass flow rate at section 2. i.e., mass flow rate is constant.
- Similarly $G = \rho_1 g A_1 V_1 = \rho_2 g A_2 V_2$
- Assuming incompressible fluid, $\rho_1 = \rho_2 = \rho$

$$A_1 V_1 = A_2 V_2 \Rightarrow Q_1 = Q_2 \Rightarrow Q_1 = Q_2 = Q_3 = Q_4$$

Continuity Equation

- Therefore, according to mass conservation for steady flow of incompressible fluids volume flow rate remains same from section to section.

Problem 5.1 *The diameters of a pipe at the sections 1 and 2 are 10 cm and 15 cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 is 5 m/s. Determine also the velocity at section 2.*

Solution. Given :

At section 1, $D_1 = 10 \text{ cm} = 0.1 \text{ m}$

$$A_1 = \frac{\pi}{4}(D_1^2) = \frac{\pi}{4}(.1)^2 = 0.007854 \text{ m}^2$$

$V_1 = 5 \text{ m/s}.$

At section 2, $D_2 = 15 \text{ cm} = 0.15 \text{ m}$

$$A_2 = \frac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$$

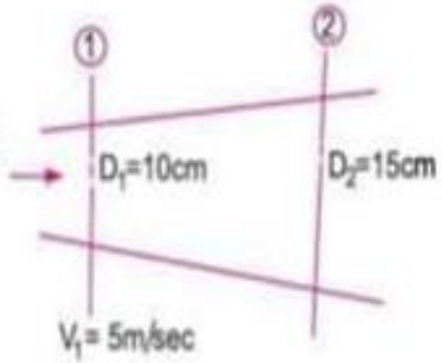


Fig. 5.2

(*i*) Discharge through pipe is given by equation (5.1)

or $Q = A_1 \times V_1$

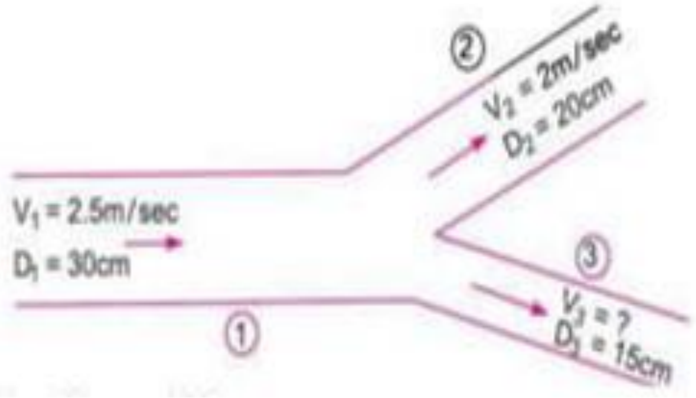
$= 0.007854 \times 5 =$ **0.03927 m³/s. Ans.**

Using equation (5.3), we have $A_1V_1 = A_2V_2$

(*ii*) $\therefore$ $V_2 = \frac{A_1V_1}{A_2} = \frac{0.007854}{0.01767} \times 5.0 =$ **2.22 m/s. Ans.**

Problem 5.2 *A 30 cm diameter pipe, conveying water, branches into two pipes of diameters 20 cm and 15 cm respectively. If the average velocity in the 30 cm diameter pipe is 2.5 m/s, find the discharge in this pipe. Also determine the velocity in 15 cm pipe if the average velocity in 20 cm diameter pipe is 2 m/s.*

Solution. Given :



$$D_1 = 30 \text{ cm} = 0.30 \text{ m}$$

$$\therefore \quad A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times .3^2 = 0.07068 \text{ m}^2$$

$$V_1 = 2.5 \text{ m/s}$$

$$D_2 = 20 \text{ cm} = 0.20 \text{ m}$$

$$\therefore \quad A_2 = \frac{\pi}{4} (.2)^2 = \frac{\pi}{4} \times .4 = 0.0314 \text{ m}^2,$$

$$V_2 = 2 \text{ m/s}$$

$$D_3 = 15 \text{ cm} = 0.15 \text{ m}$$

$$\therefore \quad A_3 = \frac{\pi}{4} (.15)^2 = \frac{\pi}{4} \times 0.225 = 0.01767 \text{ m}^2$$

Find (*i*) Discharge in pipe 1 or Q_1

(*ii*) Velocity in pipe of dia. 15 cm or V_3

Let Q_1, Q_2 and Q_3 are discharges in pipe 1, 2 and 3 respectively.

Then according to continuity equation

$$Q_1 = Q_2 + Q_3 \quad \text{...(1)}$$

(*i*) **The discharge $\mathbf{Q_1}$** in pipe 1 is given by

$$Q_1 = A_1 V_1 = 0.07068 \times 2.5 \text{ m}^3/\text{s} = \mathbf{0.1767 \text{ m}^3/\text{s}. \text{ Ans.}}$$

(*ii*) **Value of $\mathbf{V_3}$**

$$Q_2 = A_2 V_2 = 0.0314 \times 2.0 = 0.0628 \text{ m}^3/\text{s}$$

Substituting the values of Q_1 and Q_2 in equation (1)

$$0.1767 = 0.0628 + Q_3$$

$\therefore$ $$Q_3 = 0.1767 - 0.0628 = 0.1139 \text{ m}^3/\text{s}$$

But $$Q_3 = A_3 \times V_3 = 0.01767 \times V_3 \quad \text{or} \quad 0.1139 = 0.01767 \times V_3$$

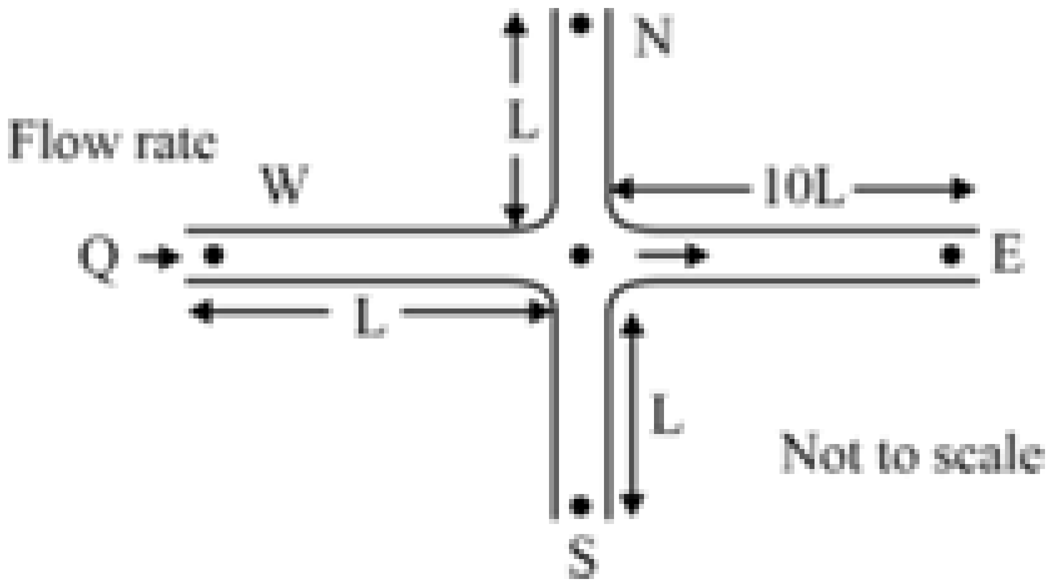
$\therefore$ $$V_3 = \frac{0.1139}{0.01767} = \mathbf{6.44 \text{ m/s. Ans.}}$$

[MCQ-2]

Q.33. In the pipe network as shown in figure, all pipes have the same cross section areas and can be assumed to have the same friction factor. The pipes connecting points W, N and S with the joint J have an equal length L. The pipe connecting points J and E has a length 10L. The pressures at the ends N, E and S are equal. The flow rate in the pipe connecting W and J is Q. Assume that the fluid flow is steady, incompressible, and the pressure losses at the pipe entrance and the junction are negligible, Consider the following statements.

I. The flow rate in pipe connecting J & E is $\frac{Q}{21}$.

II. The pressure difference between J & N is equal to the pressure difference between J & E.



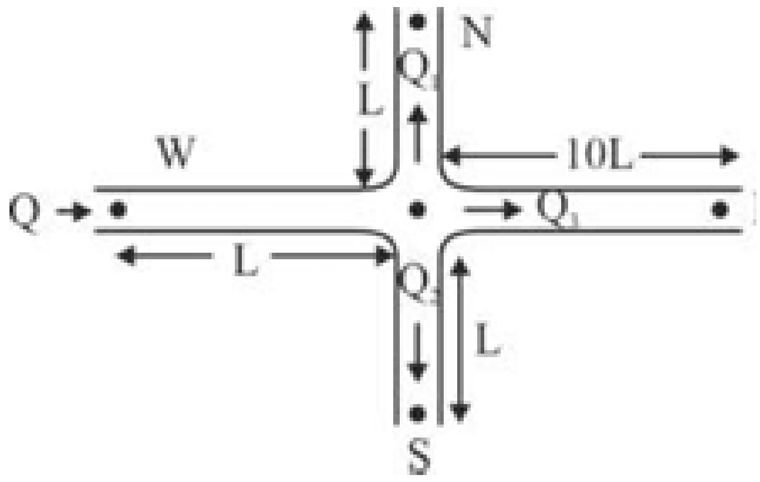
Sol. **(d)**

$Q_1 + Q_2 + Q_3 = Q$ .... (i)

$Q_1 = Q_2$ .... (ii)

From equation (i) and (ii)

$2Q_1 + Q_3 = Q$ ....(A)



$P_J - P_N = P_J - P_E$

Now $\frac{P_J - P_N}{\rho g} = \frac{P_J - P_E}{\rho g}$

$(h_L)_{JN} = (h_L)_{JE}$

$\frac{fL_{JN}Q_1^2}{12.1\ D^5} = \frac{fL_{JE}Q_3^2}{12.1\ D^5}$

$LQ_1^2 = 10LQ_3^2$

$Q_1 = \sqrt{10}Q_3$

From equation (A)

$2Q_1 + Q_3 = Q$

$2\sqrt{10}Q_3 + Q_3 = Q$

$Q_3 = \frac{Q}{1+2\sqrt{10}}$

$Q_3 = 0.136\ Q$

So statement 1 is wrong and statement 2 is correct.

Velocity of Fluid Particle

Velocity of a fluid along any direction can be defined as the rate of change of displacement of the fluid along that direction

Resultant velocity $\bar{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$$|V| = \sqrt{u^2 + v^2 + w^2}$$

Acceleration of Fluid Particle

- Acceleration of a fluid element along any direction can be defined as the rate of change of velocity of the fluid along that direction.

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$|a| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Where a_x, a_y, a_z are the acceleration vectors in X, Y and Z directions, respectively.

- a_x, a_y and a_z are termed as the total acceleration on respective directions.
- Total acceleration has two components – w.r.t time and w.r.t space and can be expressed in terms of u, v and w and this can be represented as:

$$
\begin{aligned}
a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\
a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\
a_z &= \underbrace{\frac{\partial w}{\partial t}}_{\text{local acceleration or temporal acceleration}} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{convective acceleration}}
\end{aligned}
$$

local acceleration
or
temporal acceleration

convective acceleration

- Euler equations are applicable to compressible, incompressible, non-viscous in steady or unsteady state of flow.

Let V is the resultant velocity at any point in a fluid flow. Let u, v and w are its component in x, y and z directions. The velocity components are functions of space-co-ordinates and time. Mathematically, the velocity components are given as

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

and Resultant velocity, $V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x, a_y and a_z are the **total acceleration** in x, y and z directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But $\frac{dx}{dt} = u, \frac{dy}{dt} = v \text{ and } \frac{dz}{dt} = w$

$\therefore$

$$\left.\begin{aligned} a_x &= \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \\ a_y &= \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\ a_z &= \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \end{aligned}\right] \quad \text{...(5.6)}$$

Similarly,

Local Acceleration

For steady flow, $\frac{\partial V}{\partial t} = 0$, where V is resultant velocity

or $$\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0 \text{ and } \frac{\partial w}{\partial t} = 0$$

Hence acceleration in x, y and z directions becomes

$$\left.\begin{aligned} a_x &= \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right] \quad \ldots(5.7)$$

Acceleration vector
$$\left.\begin{aligned} A &= a_x i + a_y j + a_z k \\ &= \sqrt{a_x^2 + a_y^2 + a_z^2}\,. \end{aligned}\right] \quad \ldots(5.8)$$

Problem 5.6 *The velocity vector in a fluid flow is given*

$$V = 4x^3 i - 10x^2 y j + 2t k.$$

Find the velocity and acceleration of a fluid particle at (2, 1, 3) at time t = 1.

Solution. The velocity components u, v and w are $u = 4x^3$, $v = -10x^2 y$, $w = 2t$

For the point (2, 1, 3), we have $x = 2$, $y = 1$ and $z = 3$ at time $t = 1$.

Hence velocity components at (2, 1, 3) are

$$u = 4 \times (2)^3 = 32 \text{ units}$$

$$v = -10(2)^2(1) = -40 \text{ units}$$

$$w = 2 \times 1 = 2 \text{ units}$$

$\therefore$ Velocity vector V at (2, 1, 3) = $32i - 40j + 2k$

or Resultant velocity $= \sqrt{u^2 + v^2 + w^2}$

$$= \sqrt{32^2 + (-40)^2 + 2^2} = \sqrt{1024 + 1600 + 4} = \mathbf{51.26 \text{ units. Ans.}}$$

Acceleration is given by equation (5.6)

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Now from velocity components, we have

$$\frac{\partial u}{\partial x} = 12x^2, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = -20xy, \frac{\partial v}{\partial y} = -10x^2, \frac{\partial v}{\partial z} = 0\frac{\partial v}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = 0, \frac{\partial w}{\partial y} = 0, \frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 2.1$$

Substituting the values, the acceleration components at (2, 1, 3) at time $t = 1$ are

$$a_x = 4x^3\,(12x^2) + (-\,10x^2y)\,(0) + 2t \times (0) + 0$$

$$= 48x^5 = 48 \times (2)^5 = 48 \times 32 = 1536 \text{ units}$$

$$a_y = 4x^3\,(-\,20xy) + (-\,10x^2y)\,(-\,10x^2) + 2t\,(0) + 0$$

$$= -\,80x^4y + 100x^4y$$

$$= -\,80\,(2)^4\,(1) + 100\,(2)^4 \times 1 = -\,1280 + 1600 = 320 \text{ units.}$$

$$a_z = 4x^3\,(0) + (-\,10x^2y)\,(0) + (2t)\,(0) + 2.1 = 2.0 \text{ units}$$

∴ Acceleration is $A = a_x i + a_y j + a_z k = \mathbf{1536i + 320j + 2k}$. **Ans.**

or Resultant $A = \sqrt{(1536)^2 + (320)^2 + (2)^2}$ units

$$= \sqrt{2359296 + 102400 + 4} = \mathbf{1568.9 \text{ units.}}$$ **Ans.**

Continuity Equation in Three Dimension :

When fluid flow through a full pipe, the volume of fluid entering in to the pipe must be equal to the volume of the fluid leaving the pipe, even if the diameter of the pipe vary.

Therefore we can define the continuity equation as the equation based on the principle of conservation of mass.

Therefore, for a flowing fluid through the pipe at every cross-section, the quantity of fluid per second will be constant.

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z directions respectively. Mass of fluid entering the face $ABCD$ per second

$$= \rho \times \text{Velocity in } x\text{-direction} \times \text{Area of } ABCD$$
$$= \rho \times u \times (dy \times dz)$$

Then mass of fluid leaving the face $EFGH$ per second $= \rho u \, dydz + \dfrac{\partial}{\partial x}(\rho u \, dydz)\, dx$

∴ Gain of mass in x-direction

$$= \text{Mass through } ABCD - \text{Mass through } EFGH \text{ per second}$$
$$= \rho u \, dydz - \rho u \, dydz - \frac{\partial}{\partial x}(\rho u \, dydz) dx$$
$$= -\frac{\partial}{\partial x}(\rho u \, dydz)\, dx$$
$$= -\frac{\partial}{\partial x}(\rho u)\, dx\, dydz \qquad \{\because dydz \text{ is constant}\}$$

Similarly, the net gain of mass in y-direction

$$= -\frac{\partial}{\partial y}(\rho v)\, dxdydz$$

and in z-direction

$$= -\frac{\partial}{\partial z}(\rho w)\, dxdydz$$

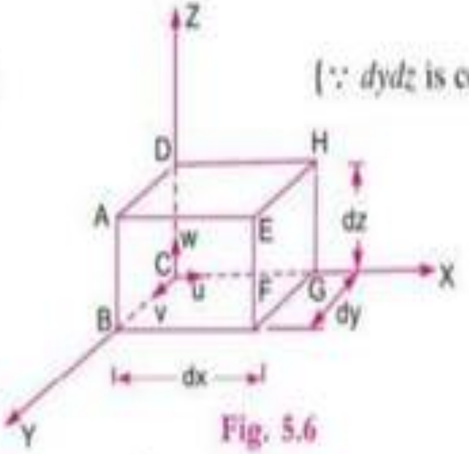


Fig. 5.6

∴ Net gain of masses $= -\left[\dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) + \dfrac{\partial}{\partial z}(\rho w)\right] dxdydz$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass of fluid in the element is $\rho . dx . dy . dz$ and its rate of increase with time is $\frac{\partial}{\partial t}(\rho \, dx . dy . dz)$ or

$$\frac{\partial \rho}{\partial t} . dx \, dy \, dz.$$

Equating the two expressions,

or
$$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t} . dxdydz$$

or
$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0 \text{ [Cancelling } dx.dy.dz \text{ from both sides]} \quad ...(5.3A)$$

Equation (5.3*A*) is the continuity equation in cartesian co-ordinates in its most general form. This equation is applicable to :

(*i*) Steady and unsteady flow,

(*ii*) Uniform and non-uniform flow, and

(*iii*) Compressible and incompressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.3*A*) becomes as

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad \text{...(5.3B)}$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{...(5.4)}$$

Equation (5.4) is the continuity equation in three-dimensions. For a two-dimensional flow, the component $w = 0$ and hence continuity equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad \text{...(5.5)}$$

Problem 5.7 *The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation :*

(i) $u = x^2 + y^2 + z^2$; $v = xy^2 - yz^2 + xy$

(ii) $v = 2y^2$, $w = 2xyz$.

Solution. The continuity equation for incompressible fluid is given by equation (5.4) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Case I.

$$u = x^2 + y^2 + z^2 \qquad \therefore \quad \frac{\partial u}{\partial x} = 2x$$

$$v = xy^2 - yz^2 + xy \qquad \therefore \quad \frac{\partial v}{\partial y} = 2xy - z^2 + x$$

Substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ in continuity equation.

$$2x + 2xy - z^2 + x + \frac{\partial w}{\partial z} = 0$$

or

$$\frac{\partial w}{\partial z} = -3x - 2xy + z^2 \text{ or } \partial w = (-3x - 2xy + z^2)\, \partial z$$

Integration of both sides gives $\int dw = \int\left(-3x - 2xy + z^2\right) dz$

or $$w = \left(-3xz - 2xyz + \frac{z^3}{3}\right) + \text{Constant of integration,}$$

where constant of integration cannot be a function of z. But it can be a function of x and y that is $f(x, y)$.

$$\therefore \quad w = \left(-3xz - 2xyz + \frac{z^3}{3}\right) + f(x, y). \textbf{ Ans.}$$

Case II. $$v = 2y^2 \quad \therefore \quad \frac{\partial v}{\partial y} = 4y$$

$$w = 2xyz \quad \therefore \quad \frac{\partial w}{\partial z} = 2xy$$

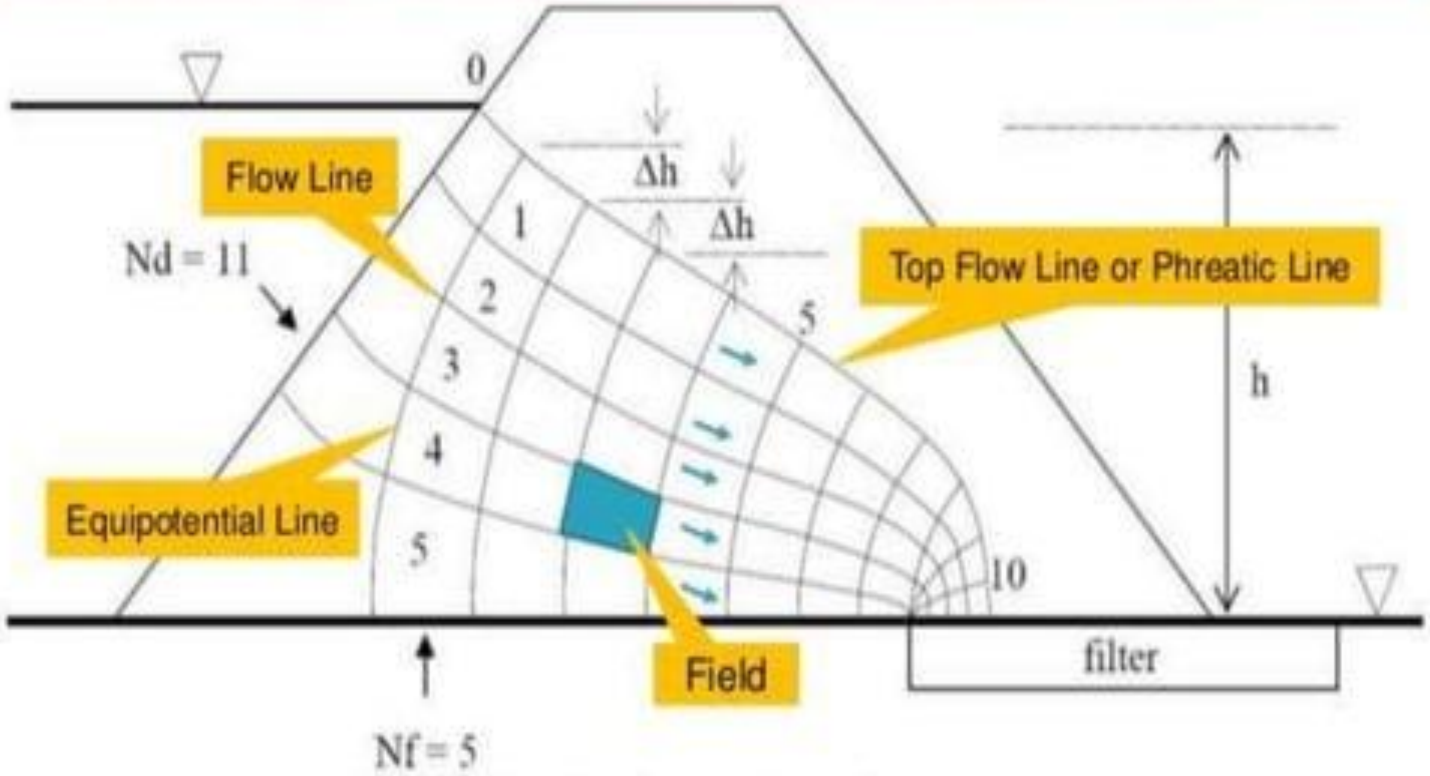
Substituting the values of $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ in continuity equation, we get

$$\frac{\partial u}{\partial x} + 4y + 2xy = 0$$

or $$\frac{\partial u}{\partial x} = -4y - 2xy \text{ or } du = (-4y - 2xy)\, dx$$

Integrating, we get $$u = -4xy - 2y\frac{x^2}{2} + f(y, z) = -4xy - x^2y + f(y, z). \textbf{ Ans.}$$

Phreatic Line is a seepage line separating saturated and unsaturated zones



Flow Net for an Earth Dam

Stream Function and Velocity Potential Function

5.8.1 Velocity Potential Function. It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi). Mathematically, the velocity, potential is defined as $\phi = f(x, y, z)$ for steady flow such that

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\} \qquad \text{...(5.9)}$$

where u, v and w are the components of velocity in x, y and z directions respectively.

The continuity equation for an incompressible steady flow is $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$

Substituting the values of u, v and w from equation (5.9), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right)=0$$

or

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0. \qquad \text{...(5.10)}$$

Equation (5.10) is a Laplace equation.

For two-dimension case, equation (5.10) reduces to $\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0.$ ...(5.11)

If any value of ϕ that satisfies the Laplace equation, will correspond to some case of fluid flow.

Properties of the Potential Function. The rotational components* are given by

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

Substituting the values, of u, v and w from equation (5.9) in the above rotational components, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \phi}{\partial x \partial z}\right]$$

and
$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y}\right]$$

If ϕ is a continuous function, then $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$; $\frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z}$; etc.

$\therefore$
$$\omega_z = \omega_y = \omega_x = 0.$$

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are :

1. If velocity potential (ϕ) exists, the flow should be irrotational.

2. If velocity potential (ϕ) satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

Problem 5.11 *The velocity potential function is given by $\phi = 5\,(x^2 - y^2)$. Calculate the velocity components at the point (4, 5).*

Solution.

$$\phi = 5\,(x^2 - y^2)$$

$$\therefore \qquad \frac{\partial \phi}{\partial x} = 10x$$

$$\frac{\partial \phi}{\partial y} = -\,10y.$$

But velocity components u and v are given by equation (5.9) as

$$u = -\frac{\partial \phi}{\partial x} = -\,10x$$

$$v = -\frac{\partial \phi}{\partial y} = -\,(-\,10y) = 10y$$

The velocity components at the point (4, 5), *i.e.*, at $x = 4$, $y = 5$

$u = -\,10 \times 4 =$ **– 40 units. Ans.**

$v = 10 \times 5 =$ **50 units. Ans.**

5.8.2 Stream Function. It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (*Psi*) and defined only for two-dimensional flow. Mathematically, for steady flow it is defined as $\psi = f(x, y)$ such that

$$\left.\begin{aligned} \frac{\partial \psi}{\partial x} &= v \\ \text{and} \qquad \frac{\partial \psi}{\partial y} &= -u \end{aligned}\right\} \qquad \text{...(5.12)}$$

The velocity components in cylindrical polar co-ordinates in terms of stream function are given as

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \text{ and } u_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{...(5.12A)}$$

where u_r = radial velocity and u_θ = tangential velocity

The continuity equation for two-dimensional flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

Substituting the values of u and v from equation (5.12), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=0 \text{ or } -\frac{\partial^2 \psi}{\partial x \partial y}+\frac{\partial^2 \psi}{\partial x \partial y}=0.$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component ω_z is given by $\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$.

Substituting the values of u and v from equation (5.12) in the above rotational component, we get

$$\omega_z=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right]=\frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}\right]$$

For irrotational flow, $\omega_z = 0$. Hence above equation becomes as $\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial \psi}{\partial y^2}=0$

which is Laplace equation for ψ.

The **properties** of stream function (ψ) are :

1. If stream function (ψ) exists, it is a possible case of fluid flow which may be rotational or irrotational.

2. If stream function (ψ) satisfies the Laplace equation, it is a possible case of an irrotational flow.

5.8.6 Relation between Stream Function and Velocity Potential Function

From equation (5.9),

we have $$u = -\frac{\partial \phi}{\partial x} \text{ and } v = -\frac{\partial \phi}{\partial y}$$

From equation (5.12), we have $u = -\dfrac{\partial \psi}{\partial y}$ and $v = \dfrac{\partial \psi}{\partial x}$

Thus, we have $$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

Hence $$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \text{and} \quad \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \quad ...(5.15)$$

Problem 5.12 *A stream function is given by* $\psi = 5x - 6y$.
Calculate the velocity components and also magnitude and direction of the resultant velocity at any point.

Solution. $\psi = 5x - 6y$

$$\therefore \quad \frac{\partial \psi}{\partial x} = 5 \text{ and } \frac{\partial \psi}{\partial y} = -6.$$

But the velocity components u and v in terms of stream function are given by equation (5.12) as

$$u = -\frac{\partial \psi}{\partial y} = -(-6) = \mathbf{6 \text{ units/sec. Ans.}}$$

$$v = \frac{\partial \psi}{\partial x} = \mathbf{5 \text{ units/sec. Ans.}}$$

Resultant velocity $= \sqrt{u^2 + v^2} = \sqrt{6^2 + 5^2} = \sqrt{36 + 25} = \sqrt{61} = 7.81$ unit/sec

Direction is given by, $\tan \theta = \frac{v}{u} = \frac{5}{6} = 0.833$

$$\therefore \quad \theta = \tan^{-1} .833 = \mathbf{39° \ 48'. \text{ Ans.}}$$

CONTINUITY EQUATION IN THREE-DIMENSIONS

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z.

Let u, v and w are the inlet velocity components in x, y and z directions respectively.

Mass of fluid entering the face ABCD per second

= ρ x Velocity in x-direction x Area of ABCD

= p x u x (dy x dz)

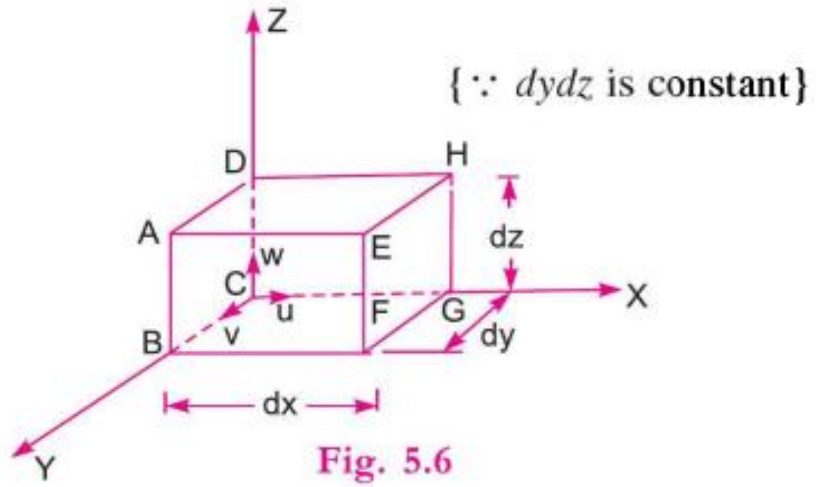


Fig. 5.6

Gain of mass in X-Direction

= Mass through *ABCD* – Mass through *EFGH* per second

$$= \rho u \, dydz - \rho u \, dydz - \frac{\partial}{\partial x}(\rho u \, dydz)dx$$

$$= -\frac{\partial}{\partial x}(\rho u \, dydz)\, dx$$

$$= -\frac{\partial}{\partial x}(\rho u)\, dx \, dydz$$

{$\because$ $dydz$ is constant}

Similarly, the net gain of mass in y-direction

$$= -\frac{\partial}{\partial y}(\rho v)\, dxdydz$$

and in z-direction

$$= -\frac{\partial}{\partial z}(\rho w)\, dxdydz$$

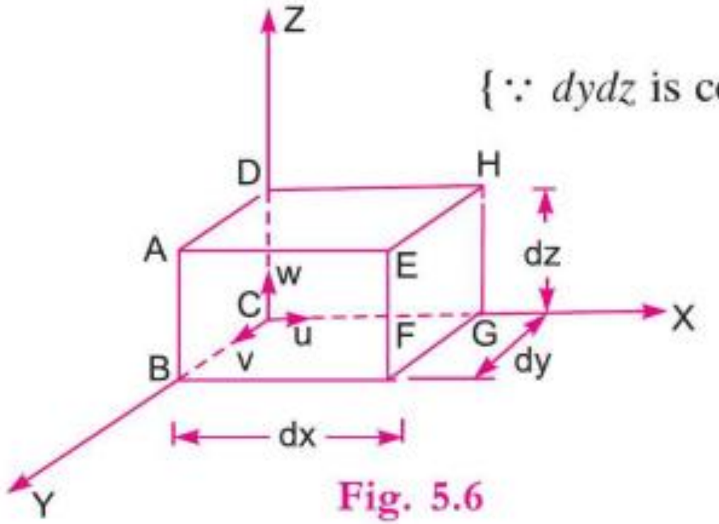


Fig. 5.6

$$\text{Net gain of masses} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz$$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass

of fluid in the element is $\rho.\, dx.\, dy.\, dz$ and its rate of increase with time is $\frac{\partial}{\partial t}(\rho\, dx.\, dy.\, dz)$

Equating the two expressions,

or $$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t}.\, dxdydz$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0$$ [Cancelling $dx.dy.dz$ from both sides]

$$\boxed{\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0}$$

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

(*i*) Steady and unsteady flow,
(*ii*) Uniform and non-uniform flow, and
(*iii*) Compressible and incompressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

Three Dimensional Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Two Dimensional Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

VELOCITY AND ACCELERATION

Let V is the resultant velocity at any point in a fluid flow.

Let u, v and w are its component in x, y and z directions.

The velocity components are functions of space-co-ordinates and time.

Mathematically, the velocity components are given as

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

Resultant velocity, $V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x, a_y and a_z are the **total acceleration** in x, y and z directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But $\frac{dx}{dt} = u, \frac{dy}{dt} = v$ and $\frac{dz}{dt} = w$

$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

For steady flow, $\frac{\partial V}{\partial t} = 0$, where V is resultant velocity

$$\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0 \text{ and } \frac{\partial w}{\partial t} = 0$$

Hence the acceleration in the x,y and z direction

$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Acceleration vector $A = a_x i + a_y j + a_z k$

$$= \sqrt{a_x^{\ 2} + a_y^{\ 2} + a_z^{\ 2}}\,.$$

Local Acceleration and Convective Acceleration:

Local acceleration is defined as the rate of increase of velocity with respect to time at a given point in a flow field.

Convective acceleration is defined as the rate of change of velocity due to the change of position of fluid particles in a fluid flow.

$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Local acceleration

Problem 5.6 *The velocity vector in a fluid flow is given*

$$V = 4x^3 i - 10x^2 y j + 2t k.$$

Find the velocity and acceleration of a fluid particle at (2, 1, 3) at time t = 1.

Resultant velocity, $V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Problem 5.6 *The velocity vector in a fluid flow is given*

$$V = 4x^3 i - 10x^2 y j + 2tk.$$

Find the velocity and acceleration of a fluid particle at (2, 1, 3) at time t = 1.

Solution. The velocity components u, v and w are $u = 4x^3$, $v = -10x^2 y$, $w = 2t$

For the point (2, 1, 3), we have $x = 2$, $y = 1$ and $z = 3$ at time $t = 1$.

Hence velocity components at (2, 1, 3) are

$$u = 4 \times (2)^3 = 32 \text{ units}$$

$$v = -10(2)^2(1) = -40 \text{ units}$$

$$w = 2 \times 1 = 2 \text{ units}$$

$\therefore$ Velocity vector V at (2, 1, 3) = $32i - 40j + 2k$

$$\text{Resultant velocity} = \sqrt{u^2 + v^2 + w^2}$$

$$= \sqrt{32^2 + (-40)^2 + 2^2} = \sqrt{1024 + 1600 + 4} = \textbf{51.26 units. Ans.}$$

Now from velocity components, we have

$$\frac{\partial u}{\partial x} = 12x^2, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = -20xy, \frac{\partial v}{\partial y} = -10x^2, \frac{\partial v}{\partial z} = 0 \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = 0, \frac{\partial w}{\partial y} = 0, \frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 2.1$$

Substituting the values, the acceleration components at (2, 1, 3) at time $t = 1$ are

$$a_x = 4x^3 (12x^2) + (-10x^2y)(0) + 2t \times (0) + 0$$

$$= 48x^5 = 48 \times (2)^5 = 48 \times 32 = 1536 \text{ units}$$

$$a_y = 4x^3 (-20xy) + (-10x^2y)(-10x^2) + 2t(0) + 0$$

$$= -80x^4y + 100x^4y$$

$$= -80(2)^4(1) + 100(2)^4 \times 1 = -1280 + 1600 = 320 \text{ units.}$$

$$a_z = 4x^3(0) + (-10x^2y)(0) + (2t)(0) + 2.1 = 2.0 \text{ units}$$

∴ Acceleration is $A = a_x i + a_y j + a_z k =$ **1536i + 320j + 2k. Ans.**

Resultant $A = \sqrt{(1536)^2 + (320)^2 + (2)^2}$ units

$= \sqrt{2359296 + 102400 + 4} =$ **1568.9 units. Ans.**

Substituting the values, the acceleration components at (2, 1, 3) at time $t = 1$ are

$$a_x = 4x^3 (12x^2) + (-10x^2y)(0) + 2t \times (0) + 0$$

$$= 48x^5 = 48 \times (2)^5 = 48 \times 32 = 1536 \text{ units}$$

$$a_y = 4x^3 (-20xy) + (-10x^2y)(-10x^2) + 2t(0) + 0$$

$$= -80x^4y + 100x^4y$$

$$= -80(2)^4(1) + 100(2)^4 \times 1 = -1280 + 1600 = 320 \text{ units.}$$

$$a_z = 4x^3(0) + (-10x^2y)(0) + (2t)(0) + 2.1 = 2.0 \text{ units}$$

∴ Acceleration is $A = a_x i + a_y j + a_z k =$ **1536i + 320j + 2k. Ans.**

Resultant $A = \sqrt{(1536)^2 + (320)^2 + (2)^2}$ units

$$= \sqrt{2359296 + 102400 + 4} = \textbf{1568.9 units. Ans.}$$

Problem 5.7 *The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation :*

(i) $u = x^2 + y^2 + z^2$; $v = xy^2 - yz^2 + xy$

(ii) $v = 2y^2$, $w = 2xyz$.

Solution. The continuity equation for incompressible fluid is given by equation (5.4) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Velocity Potential function and Steam function

Velocity Potential Function. It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi). Mathematically, the velocity, potential is defined as $\phi = f(x, y, z)$ for steady flow such that

$$\left.\begin{aligned} u &= -\frac{\partial\phi}{\partial x} \\ v &= -\frac{\partial\phi}{\partial y} \\ w &= -\frac{\partial\phi}{\partial z} \end{aligned}\right\}$$

Properties of the Potential Function. The rotational components* are given by

Properties of the Potential Function. The rotational components* are given by

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \phi}{\partial x \partial z}\right]$$

$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y}\right]$$

$$\therefore \qquad \omega_z = \omega_y = \omega_x = 0.$$

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are :

1. If velocity potential (ϕ) exists, the flow should be irrotational.

2. If velocity potential (ϕ) satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

Stream Function. It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (*Psi*) and defined only for two-dimensional flow. Mathematically, for steady flow it is defined as $\psi = f(x, y)$ such that

$$\left.\begin{aligned} \frac{\partial \psi}{\partial x} &= v \\ \frac{\partial \psi}{\partial y} &= -u \end{aligned}\right\}$$

The continuity equation for two-dimensional flow is $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0.$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0 \text{ or } -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0.$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component ω_z is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right] = \frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right]$$

For irrotational flow, $\omega_z = 0$. Hence above equation becomes as $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial y^2} = 0$

The **properties** of stream function (ψ) are :

1. If stream function (ψ) exists, it is a possible case of fluid flow which may be rotational or irrotational.
2. If stream function (ψ) satisfies the Laplace equation, it is a possible case of an irrotational flow.

Relation between Stream Function and Velocity Potential Function

$$u = -\frac{\partial\phi}{\partial x} \text{ and } v = -\frac{\partial\phi}{\partial y} \qquad u = -\frac{\partial\psi}{\partial y} \text{ and } v = \frac{\partial\psi}{\partial x}$$

Thus, we have

$$u = -\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y} \text{ and } v = -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}$$

Hence

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$$

and

$$\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

DYNAMICS OF FLUID FLOW

In the previous Unit ,

we studied the velocity and acceleration at a point in a fluid flow, without taking into consideration the forces causing the flow.

This Unit includes the study of forces causing fluid flow. Thus **dynamics of fluid flow is the study of fluid motion with the forces causing flow.**

The dynamic behaviour of the fluid flow is analysed by the Newton's second law of motion, which relates the acceleration with the forces.

The fluid is assumed to be **incompressible** and **non-viscous.**

EQUATIONS OF MOTION

According to Newton's second law of motion, the net force F, acting on a fluid element in the direction of x is equal to mass m of the fluid element multiplied by the acceleration a, in the x-direction. Thus mathematically,

$$\boldsymbol{F_X} = \mathbf{M}\, \boldsymbol{a_x}$$

In the fluid flow, the following forces are present:

(i) F_g, gravity force.

(ii) F_p the pressure force.

(iii) F_v force due to viscosity.

(iv) F_t force due to turbulence.

(v) F_c, force due to compressibility.

The net force

Net Force = $\boldsymbol{F_g + F_p + F_v + F_t + F_c}$

the net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x + (F_c)_x.$$

(*i*) If the force due to compressibility, F_c is negligible, the resulting net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x$$

and equation of motions are called **Reynold's equations of motion.**

(*ii*) For flow, where (F_t) is negligible, the resulting equations of motion are known as **Navier-Stokes Equation.**

(*iii*) If the flow is assumed to be ideal, viscous force (F_v) is zero and equation of motions are known as **Euler's equation of motion.**

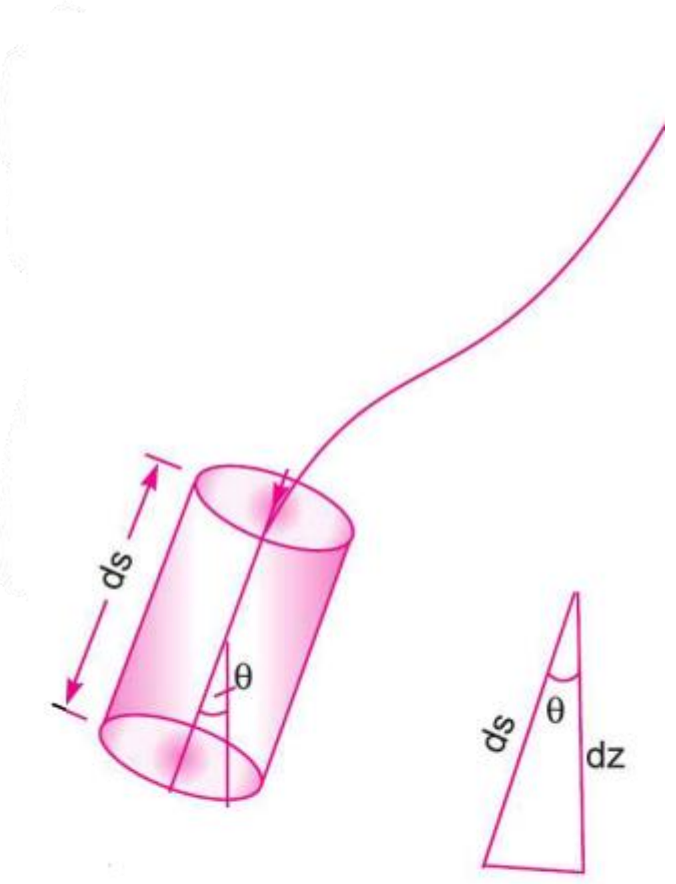
EULER'S EQUATION OF MOTION

This is equation of motion in which the forces due to gravity and pressure are taken into consideration. This is derived by considering the motion of a fluid element along a stream-line as :

Consider a stream-line in which flow is taking place in s-direction as shown in Figure.

Consider a cylindrical element of cross-section **dA** and Length **ds**.

The forces acting on the cylindrical element are:



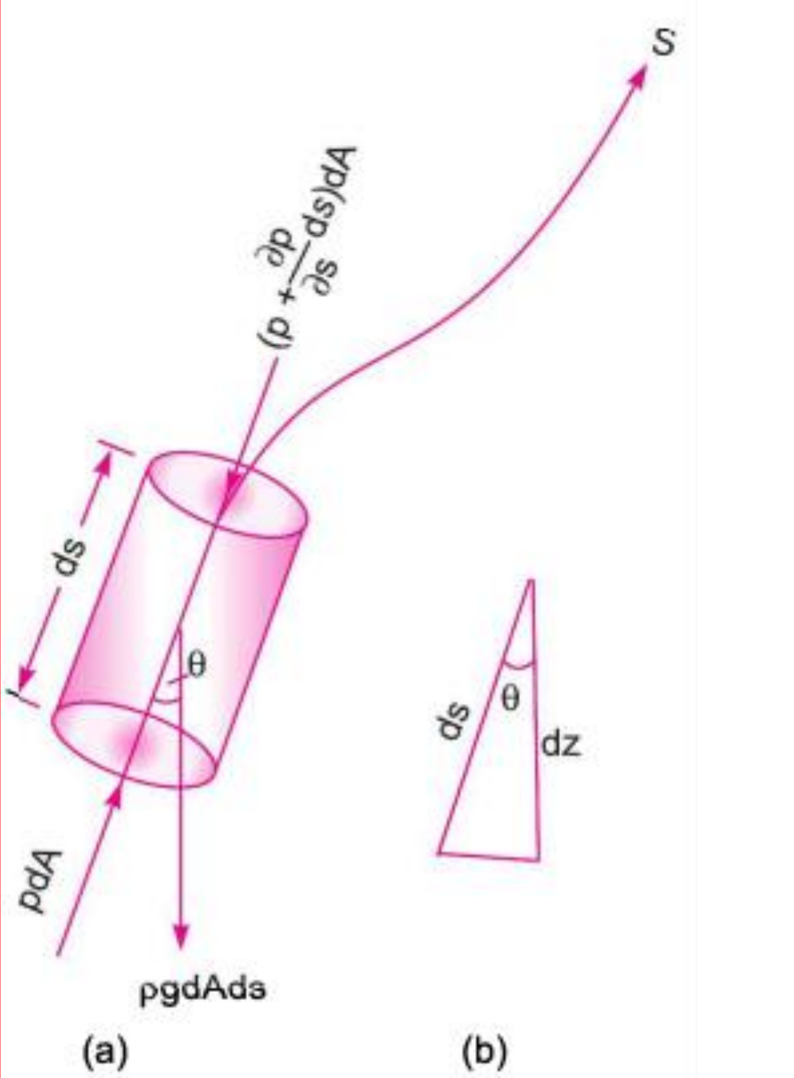
1. Pressure force pdA in the direction of flow.

2. Pressure force $\left(p + \frac{\partial p}{\partial s} ds\right) dA$ opposite to the direction of flow.

3. Weight of element $\rho g dA ds$.

Let **θ** is the angle between the direction of flow and the line of action of the weight of element.

The resultant force on the fluid element in the direction of **s** must be equal to the mass of fluid element X acceleration in the direction s.



Forces on a fluid element.

$$pdA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \rho g dA ds \cos\theta \quad = \rho dA ds \times a_s$$

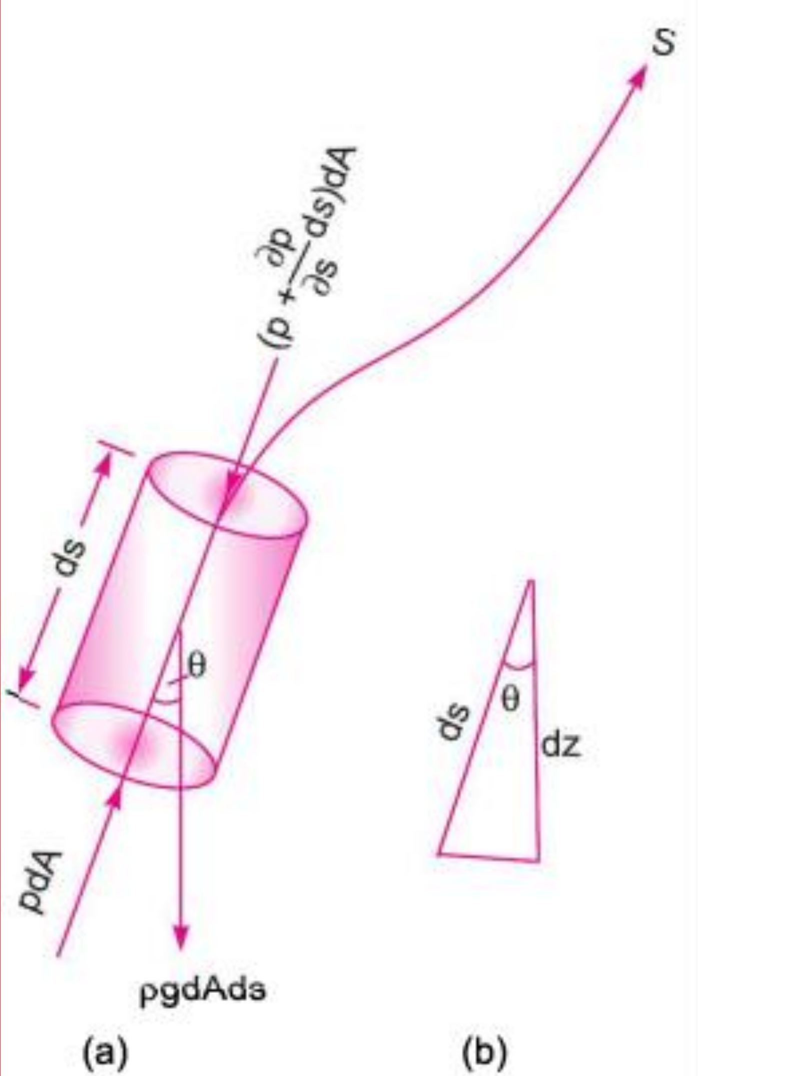
where a_s is the acceleration in the direction of s.

Now $$a_s = \frac{dv}{dt}, \text{ where } v \text{ is a function of } s \text{ and } t.$$

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v\partial v}{\partial s} + \frac{\partial v}{\partial t} \quad \left\{\because \frac{ds}{dt} = v\right\}$$

If the flow is steady, $\frac{\partial v}{\partial t} = 0$

$$\therefore \qquad a_s = \frac{v\partial v}{\partial s}$$



Forces on a fluid element.

$$pdA - \left(p + \frac{\partial p}{\partial s}\,ds\right) dA - \rho g dAds \cos\theta = \rho dAds \times a_s$$

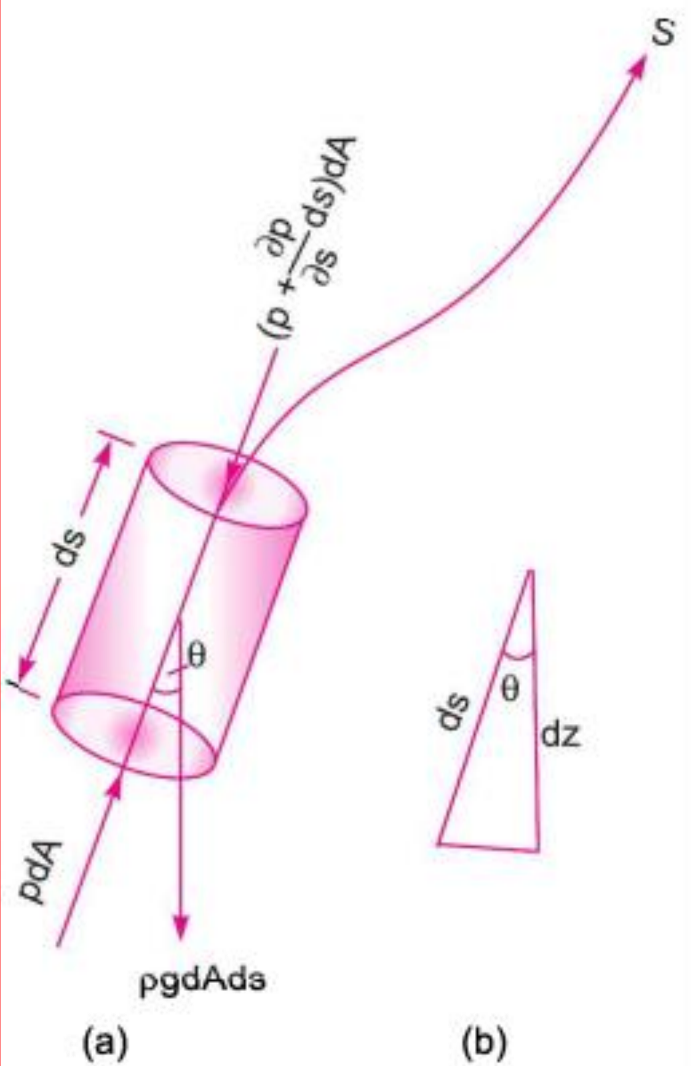
$$-\frac{\partial p}{\partial s}\,dsdA - \rho g\ dAds \cos\theta = \rho dAds \times \frac{\partial v}{\partial s}$$

Dividing by $\rho dsdA$,

$$-\frac{\partial p}{\rho \partial s} - g\cos\theta = \frac{v\partial v}{\partial s}$$

$$\frac{\partial p}{\rho \partial s} + g\cos\theta + v\frac{\partial v}{\partial s} = 0$$

we have $\cos\theta = \dfrac{dz}{ds}$



Forces on a fluid element.

$$\frac{\partial p}{\rho \partial s} + g \cos \theta + v \frac{\partial v}{\partial s} = 0$$

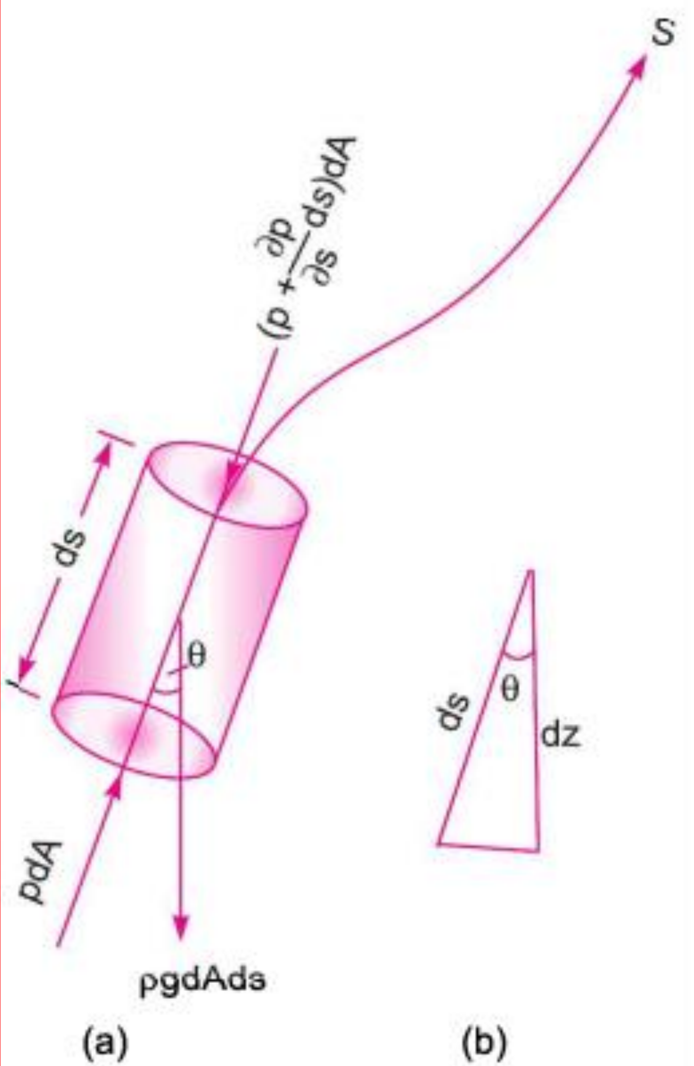
we have $\cos \theta = \frac{dz}{ds}$

$$\frac{1}{\rho} \frac{dp}{ds} + g \frac{dz}{ds} + \frac{vdv}{ds} = 0$$

$$\frac{dp}{\rho} + gdz + vdv = 0$$

$$\frac{dp}{\rho} + gdz + vdv = 0$$

Euler's equation of motion.



Forces on a fluid element.

BERNOULLI'S EQUATION FROM EULER'S EQUATION

$$\frac{dp}{\rho} + g\,dz + v\,dv = 0$$

Euler's equation of motion.

Bernoulli's equation is obtained by integrating the Euler's equation of motion

$$\int \frac{dp}{\rho} + \int g\,dz + \int v\,dv = \text{constant}$$

If flow is incompressible, ρ is constant and

$$\frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant} \qquad \frac{p}{\rho g} + z + \frac{v^2}{2g} = \text{constant} \qquad \frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant}$$

$\frac{p}{\rho g}$ = pressure energy per unit weight of fluid or pressure head.

BERNOULLI'S EQUATION FROM EULER'S EQUATION

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant}$$

$\frac{p}{\rho g}$ = pressure energy per unit weight of fluid or pressure head.

$v^2/2g$ = kinetic energy per unit weight or kinetic head.

z = potential energy per unit weight or potential head.

The following are the assumptions made in the derivation of Bernoulli's equation :

(*i*) The fluid is ideal, *i.e.*, viscosity is zero
(*ii*) The flow is steady
(*iii*) The flow is incompressible
(*iv*) The flow is irrotational.

PRACTICAL APPLICATIONS OF BERNOULLI'S EQUATION

Bernoulli's equation is applied in all problems of incompressible fluid flow where energy considerations are involved. But we shall consider its application to the following measuring devices:

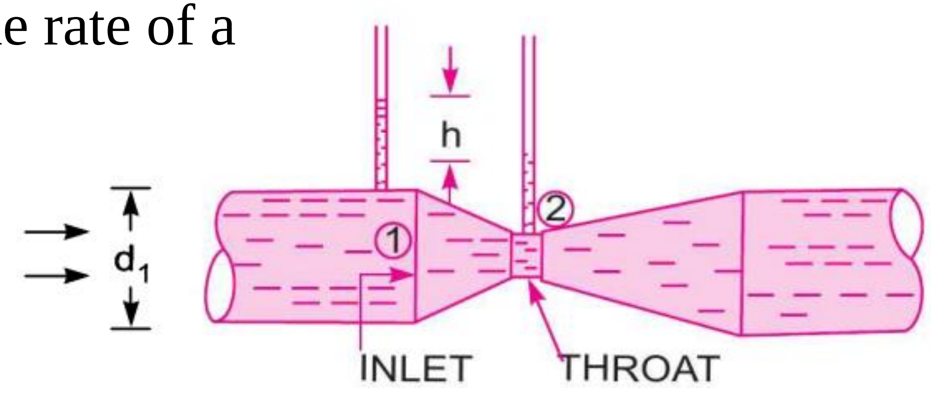
1. Venturimeter.
2. Orifice meter.
3. Pitot-tube.

1.Venturimeter. A venturimeter is a device used for measuring the rate of a flow of a fluid flowing through a pipe. It consists of three parts:

(i) A short converging part,

(ii) Throat, and

(iii) Diverging part.



Venturimeter.

It is based on the Principle of Bernoulli's equation.

Consider a venturimeter fitted in a horizontal pipe through which a fluid is flowing (say water), as shown in Figure and

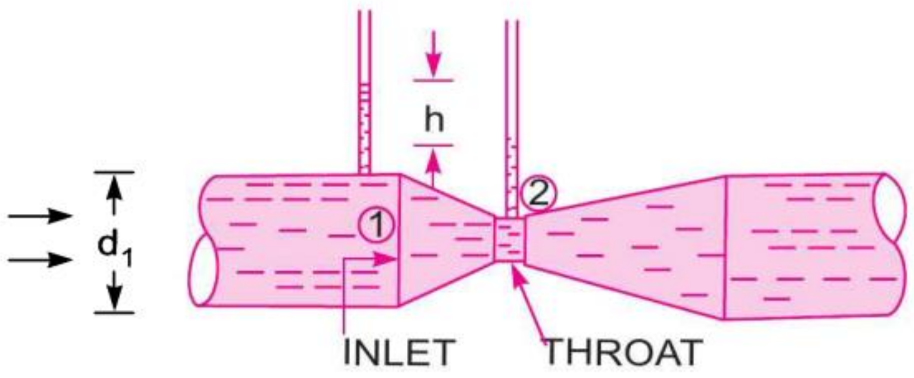
Let d_1 = diameter at inlet or at section (1),

P_1= pressure at section (1)

V_1 = velocity of fluid at section (1), a = area at section (1) = $\frac{\pi}{4} {d_1}^2$

d_2, P_2, V_2, a_2 are corresponding values at section (2).

Applying Bernoulli's equation at sections (1) and (2), we get



Venturimeter.

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

As pipe is horizontal, hence $z_1 = z_2$

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g}$$

or

$$\frac{p_1 - p_2}{\rho g} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

But $\frac{p_1 - p_2}{\rho g}$ is the difference of pressure heads at sections 1 and 2 and it is equal to h

$$\frac{p_1 - p_2}{\rho g} = h$$

$$h = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

Now applying continuity equation at sections 1 and 2

$$a_1 v_1 = a_2 v_2 \quad \text{or} \quad v_1 = \frac{a_2 v_2}{a_1}$$

$$h = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

$$h = \frac{v_2^2}{2g} - \frac{\left(\frac{a_2 v_2}{a_1}\right)^2}{2g} = \frac{v_2^2}{2g}\left[1 - \frac{a_2^2}{a_1^2}\right] = \frac{v_2^2}{2g}\left[\frac{a_1^2 - a_2^2}{a_1^2}\right]$$

$${v_2}^2 = 2gh \frac{a_1^2}{a_1^2 - a_2^2}$$

$$v_2 = \sqrt{2gh \frac{a_1^2}{a_1^2 - a_2^2}}$$

$$= \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$$

Discharge : $Q = a_2 v_2$

$$= a_2 \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$v_2 = \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$$

$$= \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

Discharge under ideal conditions and is called, theoretical discharge.

Actual discharge will be **less than** theoretical discharge.

$$Q_{act} = C_d \times \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

C_d = Co-efficient of venturimeter and its value is less than 1.

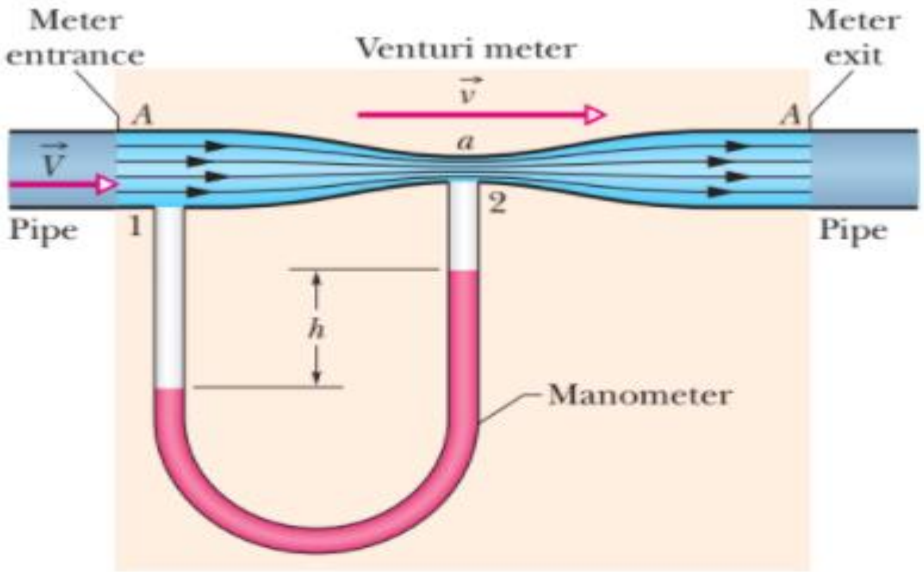
Value of 'h' given by differential U-tube manometer

Case I. Let the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe. Let

S_h = Specific gravity of the heavier liquid

S_0 = Specific gravity of the liquid flowing through pipe

x = Difference of the heavier liquid column in U-tube



$$h = x\left[\frac{S_h}{S_o} - 1\right]$$

Case II. If the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given by

where

S_l = Specific gravity of lighter liquid in U-tube

S_0 = Specific gravity of fluid flowing through pipe

x = Difference of the lighter liquid columns in U-tube.

$$h = x\left[1 - \frac{S_l}{S_o}\right]$$

Case III. Inclined Venturimeter with Differential U-tube manometer.

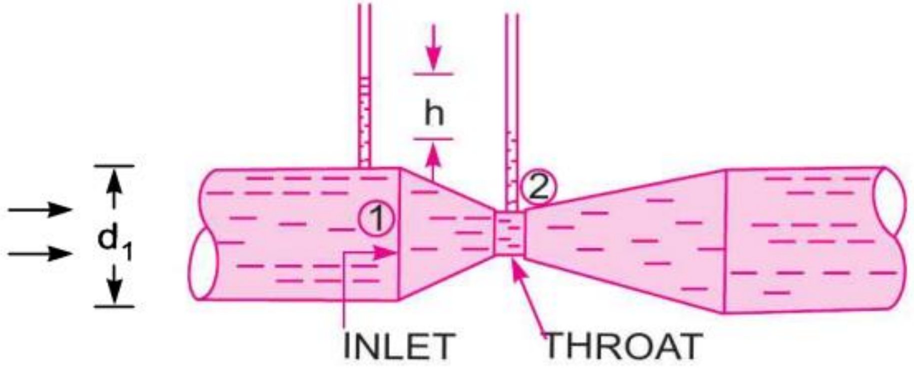
The above two cases are given for a horizontal venturimeter. This case is related to inclined venturimeter having differential U-tube manometer. Let the differential manometer contains heavier liquid then h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[\frac{S_h}{S_o} - 1\right]$$

Case IV : Similarly, for inclined venturimeter in which differential manometer contains a liquid .which is lighter than the liquid flowing through the pipe, the value of **h** is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[1 - \frac{S_l}{S_o}\right]$$

Problem 6.10 : A horizontal venturimeter with inlet and throat diameters **30** cm and **15 cm** respectively is used to measure the flow of water. The reading of differential manometer connected to the inlet and the throat is 20 cm of mercury. Determine the rate of flow. Take $\boldsymbol{C_d}$ **= 0.98.**



Venturimeter.

Diameter at inlet = d_1 = 30 cm

Area at inlet, $a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} (30)^2 = 706.85 \text{ cm}^2$

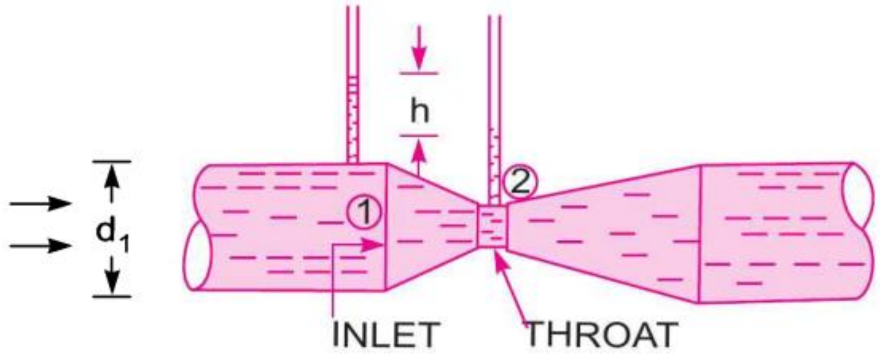
Dia. at throat, $d_2 = 15$ cm

$$a_2 = \frac{\pi}{4} \times 15^2 = 176.7 \text{ cm}^2$$

$C_d = 0.98$

Reading of differential manometer = x = 20 cm of mercury.

$$h = x\left[\frac{S_h}{S_o} - 1\right]$$



Venturimeter.

where S_h = Sp. gravity of mercury = 13.6, S_o = Sp. gravity of water = 1

$$= 20\left[\frac{13.6}{1} - 1\right] = 20 \times 12.6 \text{ cm} = 252.0 \text{ cm of water.}$$

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 9.81 \times 252}$$

$$= \frac{86067593.36}{\sqrt{499636.9 - 31222.9}} = \frac{86067593.36}{684.4}$$

$= 125756 \text{ cm}^3\text{/s} = \frac{125756}{1000}$ lit/s = **125.756 lit/s. Ans.**

Problem 6.11 An oil of specific gravity 0.8 is flowing through a venturimeter having inlet diameter 20 cm and throat diameter 10 cm. The oil-mercury differential manometer shows a reading of 25 cm. Calculate the discharge of oil through the horizontal venturimeter. Take C_d= 0.98.

Sp. gr. of oil, $S_o = 0.8$

Sp. gr. of mercury, $S_h = 13.6$

Reading of differential manometer, $x = 25$ cm

$$\therefore \text{ Difference of pressure head, } h = x\left[\frac{S_h}{S_o} - 1\right]$$

$$= 25\left[\frac{13.6}{0.8} - 1\right] \text{ cm of oil} = 25\,[17 - 1] = 400 \text{ cm of oil.}$$

$d_1 = 20$ cm $\qquad a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 20^2 = 314.16 \text{ cm}^2$

$d_2 = 10$ cm $\qquad a_2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2$

$C_d = 0.98$

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 400}$$

$$= \frac{21421375.68}{\sqrt{98696 - 6168}} = \frac{21421375.68}{304} \text{ cm}^3/\text{s}$$

$= 70465 \text{ cm}^3/\text{s} =$ **70.465 litres/s. Ans.**

Momentum Equation

In fluid Mechanics, the analysis of motion is performed in the same way as in solid mechanics (by use of "Newton's Laws of Motion").

From solid mechanics (Newton's Second Law) stated that:

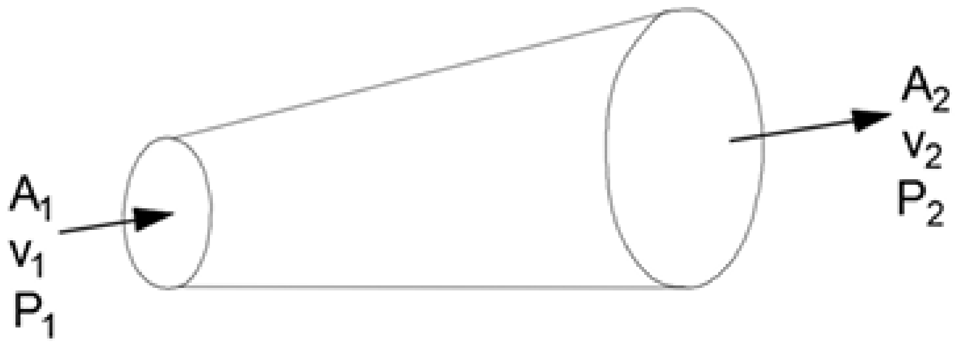
Total Force(F) = ma

m = mass of the solid body , a = acceleration

But, in fluid mechanics, it is not clear what mass of moving fluid, thus we should use a different form of the equation of Newton's Second Law.

Momentum Equation

Newton's 2nd Law (for fluids) can be written as following: The rate of change of momentum of a body is equal to the total force acting on the body, and takes place in the direction of the force.



Rate of change of momentum is:

= Mass Flow Rate × Change of velocity

= $m(V_2 - V_1)$ =

According Newton's 2nd law:

$F = m(V_2 - V_1)$,

Mass flow rate = $m = \rho Q$

$F = \rho Q\,(V_2 - V_1)$,

This is the total force acting on the fluid in the direction of motion.

Momentum Equation

Newtons Second Law

Total Force(F) = ma

$$a = \frac{dv}{dt}$$

$$(F) = m\frac{dv}{dt} = \frac{d(mv)}{dt}$$

Rate of change of momentum

F .dt = d (mv)

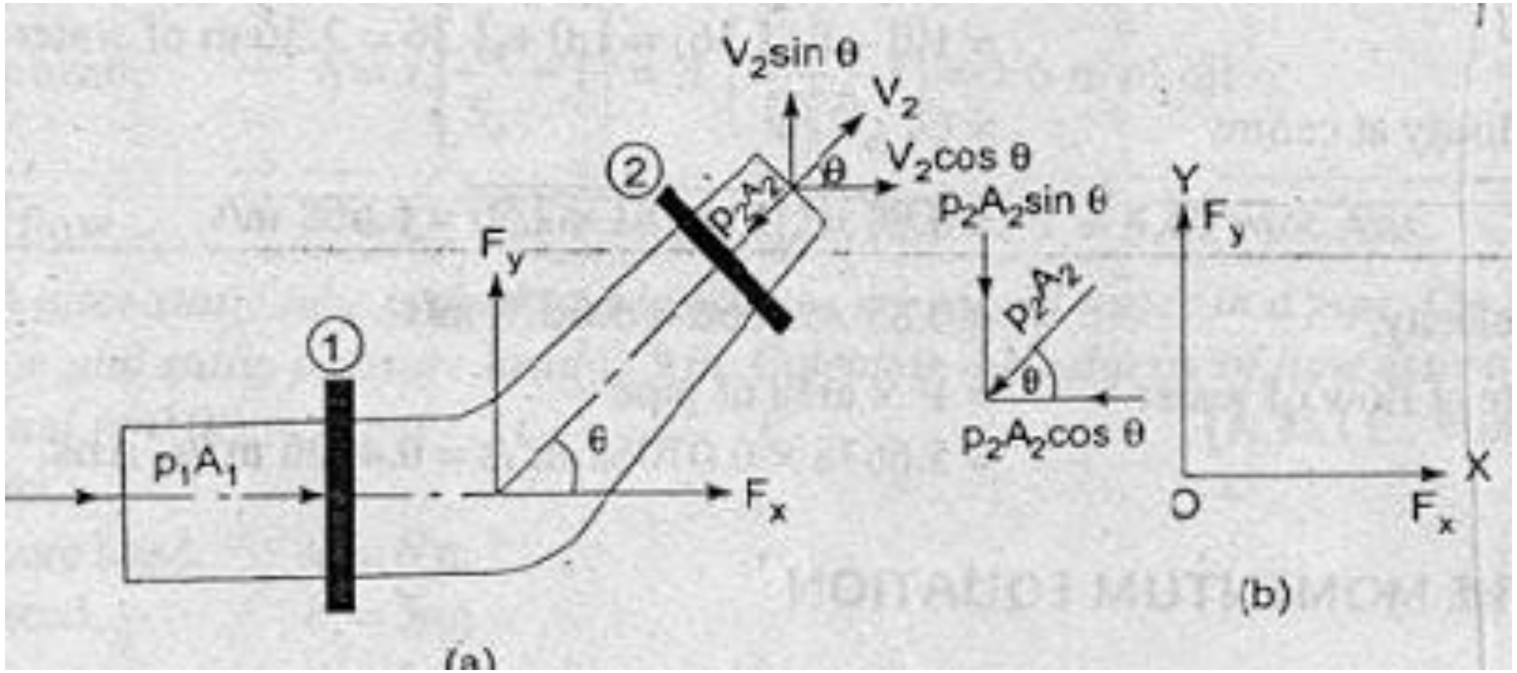
Impulse = change of momentum

Impulse momentum equation

Application of Momentum Equation

This equation has wide-ranging applications in various fields. One key application is in the **design and analysis of fluid flow systems, such as pipes, channels, and centrifugal pumps,** where the momentum equation helps determine the force and pressure exerted on the fluid as it flows through these systems.

-Pipe Bend

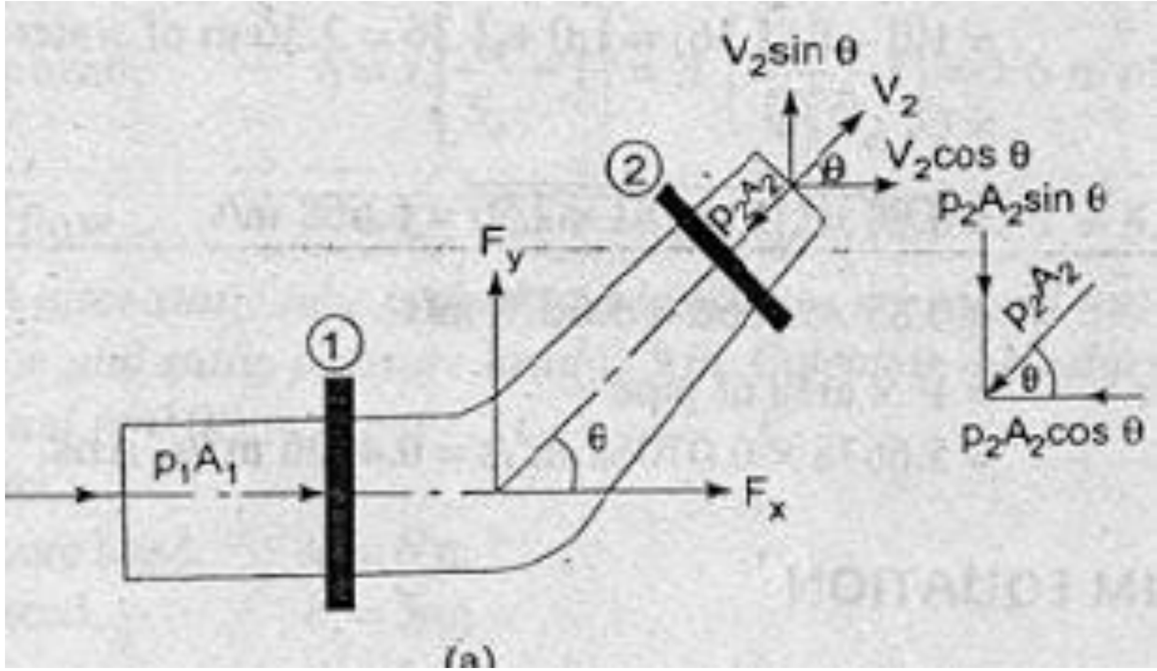


F_X = Force exerted by liquid on nozzle **or** pipe bend in X − Direction

$-F_x$ = Force exerted by Nozzle or Pipe bend on the Liquid in X-direction

F_Y = Force exerted by liquid on nozzle **or** pipe bend in Y − Direction

$-F_Y$ = Force exerted by Nozzle or Pipe bend on the Liquid in Y-direction



Force in X- Direction

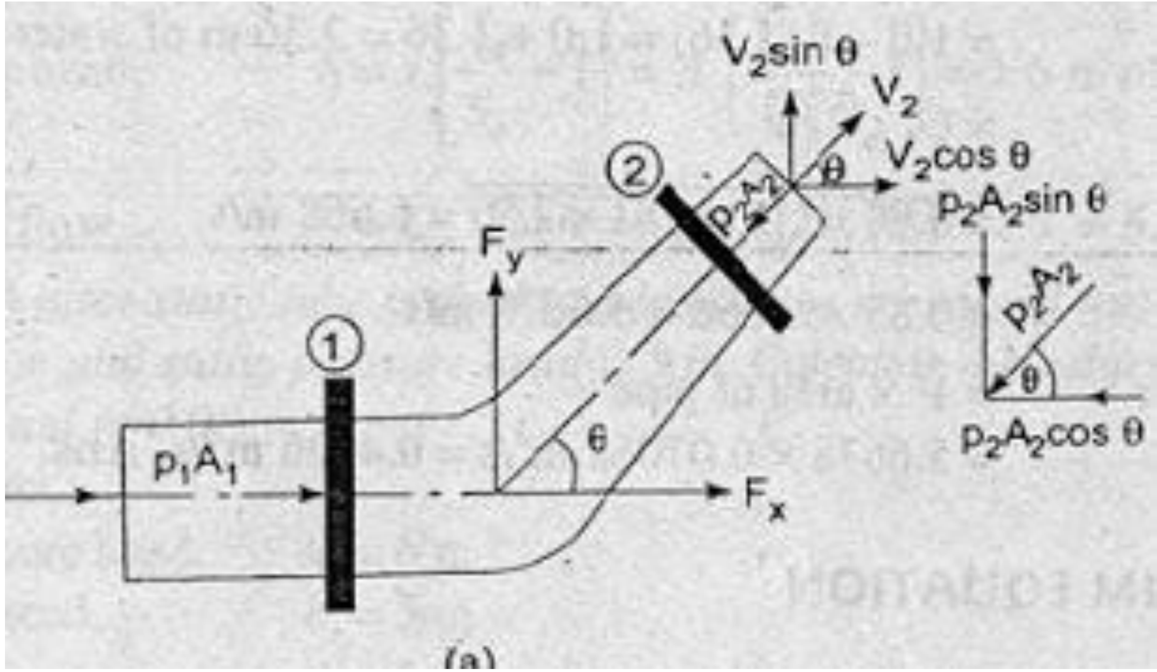
Net Force in X-Direction

$= P_1A_1 - P_2A_2 \cos\theta - F_X$

Rate of change of Momentum

$= \dot{m}\,(V_2 \cos\theta - V_1)$

$= \rho\, A_1V_1\,(V_2 \cos\theta - V_1)$



Force in Y- Direction

Net Force in Y-Direction

$= 0 - P_2A_2 \sin\theta - F_y$

Rate of change of Momentum

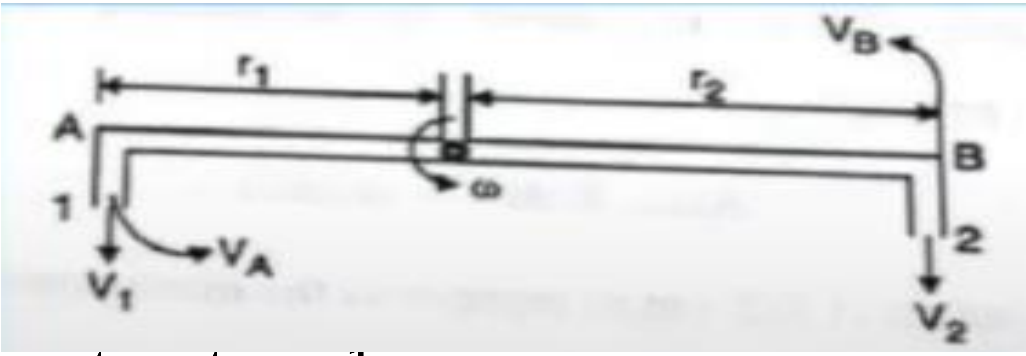
$= \dot{m}\, V_2 \sin\theta - 0$

Resultant Force $= \sqrt{F_X^2 + F_y^2}$

$\tan\theta = \frac{F_Y}{F_X}$

Momentum of Momentum Equation

Net Torque = Rate of change of momentum of momentum



Momentum at section -1

m V_1

m $V_1 r_1$

Momentum at section -2

m V_2

m $V_2 r_2$

Resulting Torque or Net Torque

(T) = $\dot{m}$ $(V_2 r_2 - V_1 r_1)$

(T) = ρ $A_1 V_1 (V_2 r_2 - V_1 r_1)$

IMPACT OF JETS AND JET PROPULSION

INTRODUCTION

The liquid comes out in the form of a jet from the outlet of a nozzle, which is fitted to a pipe through which the liquid is flowing under pressure.

If some plate, which may be fixed or moving, is placed in the path of the jet, a force is exerted by the jet on the plate. This force is obtained from Newton's second law of motion or from impulse-momentum equation.

Thus impact of jet means the force exerted by the jet on a plate which may be stationary or moving.

In this Unit, the following cases of the impact of jet i.e., the force exerted by the jet on a plate, will be considered:

1. Force exerted by the jet on a stationary plate when

(a) Plate is vertical to the jet,

(b) Plate is inclined to the jet, and

(c) Plate is curved.

2. Force exerted by the jet on a moving plate, when

(a) Plate is vertical to the jet,

(b) Plate is inclined to the jet, and

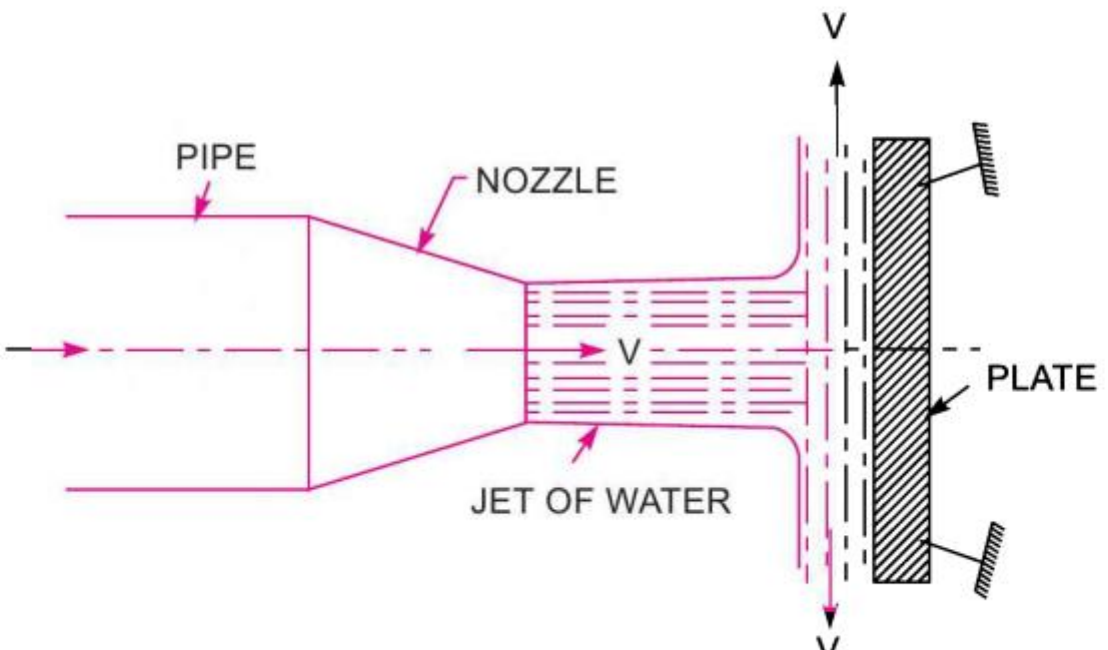
(c) Plate is curved.

1. Force exerted by the jet on a stationary Vertical plate

V = Velocity of the jet

d= Diameter of the jet

$$a = \text{area of cross-section of the jet} = \frac{\pi}{4} d^2.$$

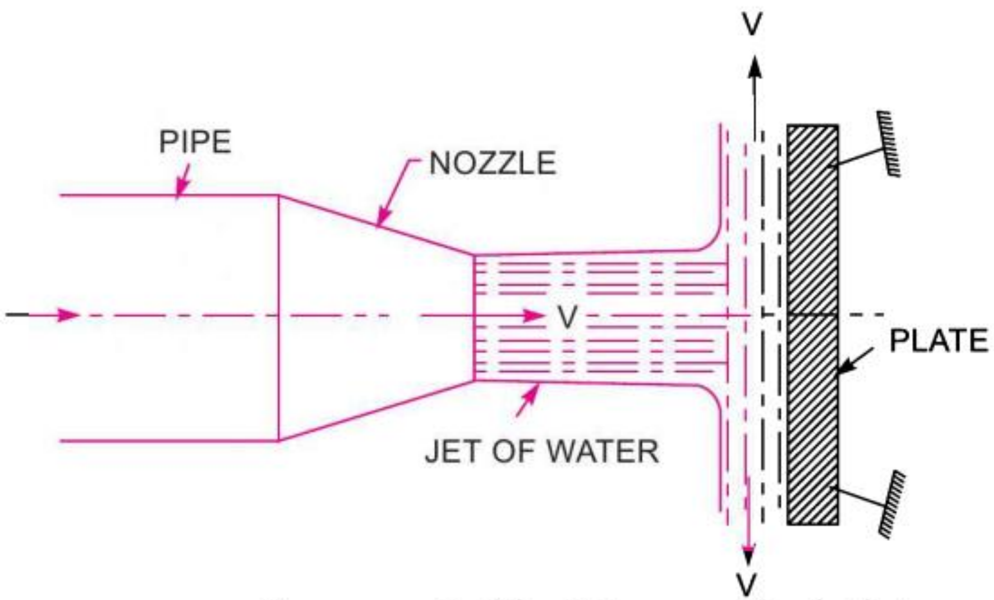


Force exerted by jet on vertical plate.

The jet after striking the plate, will move along the plate. But the plate is at right angles to the jet. Hence the jet after striking, will get deflected through 90°.

Hence the component of the velocity of jet, in the direction of jet, after striking will be **zero.**

The force exerted by the jet on the plate in the direction of jet, The force exerted by the jet on the plate in the direction of jet,



Force exerted by jet on vertical plate.

$$= \frac{\text{Initial momentum} - \text{Final momentum}}{\text{Time}}$$

$$= \frac{(\text{Mass} \times \text{Initial velocity} - \text{Mass} \times \text{Final velocity})}{\text{Time}}$$

$$= \frac{\text{Mass}}{\text{Time}} [\text{Initial velocity} - \text{Final velocity}]$$

$$= (\text{Mass/sec}) \times (\text{velocity of jet before striking} - \text{velocity of jet after striking})$$

$$= \rho a V[V - 0]$$

$$= \rho a V^2$$

For deriving above equation,

If the force exerted on the jet is to be calculated : then final velocity -initial velocity

if the force exerted by the jet on the plate is to be calculated, then initial velocity - final velocity is taken.

Force Exerted by a Jet on Stationary Inclined Flat Plate.

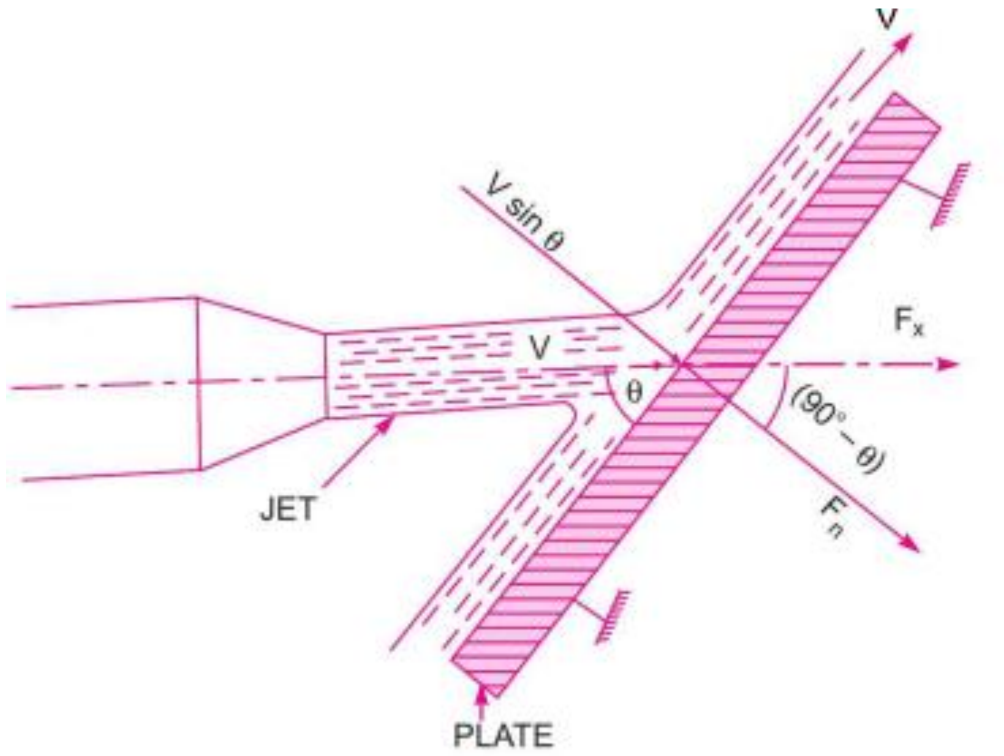
Let a jet of water, coming out from the nozzle, strikes an inclined flat plate as shown in Figure

Let V = Velocity of jet in the direction of x,

Θ = Angle between the jet and plate,

a = Area of cross-section of the jet.

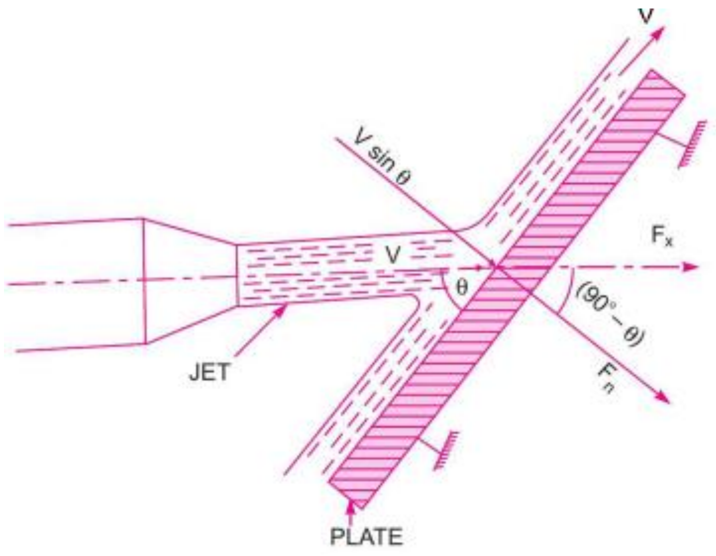
Then mass of water per sec striking the plate = ρ x aV



Jet striking stationary inclined plate

If the plate is smooth and if it is assumed that there is no loss of energy due to impact of the jet, then jet will move over the plate after striking with a velocity equal to initial velocity i.e., with a velocity V.

Let us find the force exerted by the jet on the plate in the direction normal to the plate. Let this force is represented by F_n



F_n = mass of jet striking per second × [Initial velocity of jet before striking in the direction of n - Final velocity of jet after striking in the direction of n]

$$= \rho a V\,[V \sin\theta - 0\,]$$

$$= \rho a V^2 \sin\theta$$

This force can be resolved into two components, one in the direction of the jet and other perpendicular to the direction of flow.

Then we have,

F_x = component of F_n in the direction of flow

$= F_n \cos(90° - \theta)$

$= F_n \sin\theta$

$= \rho A V^2 \sin\theta \times \sin\theta$

$= \rho A V^2 \sin^2\theta$

F_y = component of F_n, perpendicular to flow

$= F_n \sin(90° - \theta)$

$= F_n \cos\theta \quad = \rho A V^2 \sin\theta \cos\theta.$

Force Exerted by a Jet on Stationary Curved Plate

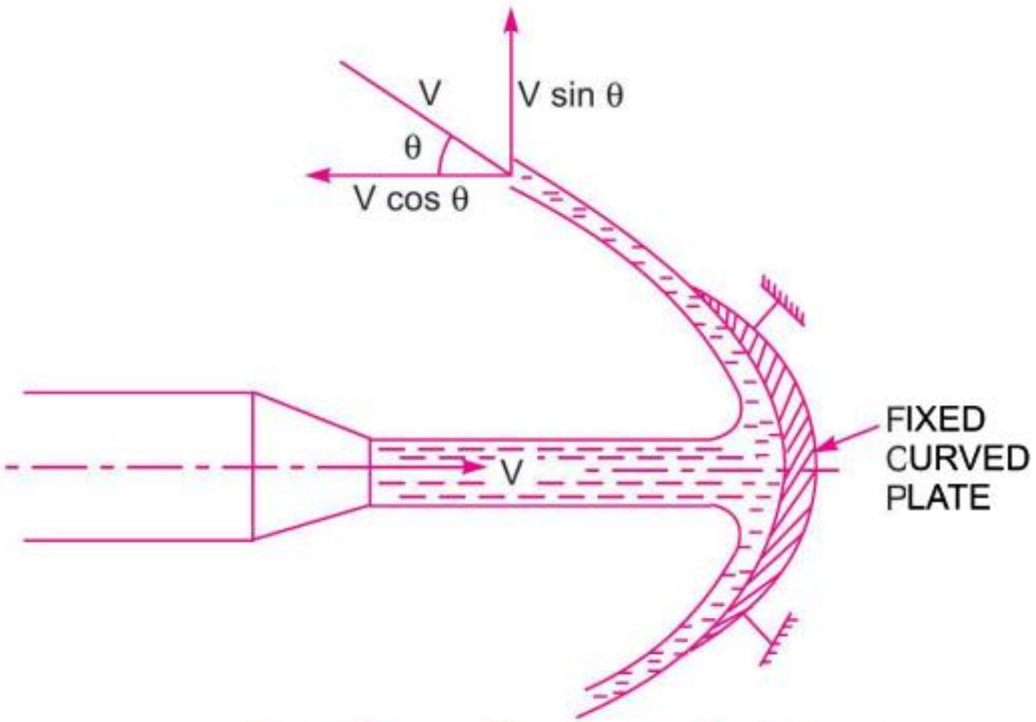
(A) Jet strikes the curved plate at the centre.

Let a jet of water strikes a fixed curved plate at the centre as shown in Figure.

The jet after striking the plate, comes out with the same velocity if the plate is smooth and there is no loss of energy due to impact of the jet, in the tangential direction of the curved plate.

The velocity at outlet of the plate can be resolved into two components, **one in the direction of jet** and **other perpendicular to the direction of the jet.**

Component of velocity in the direction of jet = - V cos Θ



Jet striking a fixed curved plate at centre.

(-ve sign is taken as the velocity at outlet is in the opposit[e]

direction of the jet of water coming out from nozzle).

Component of velocity perpendicular to the jet = V sin θ

Force exerted by the jet in the direction of jet,

$$F_x = \text{Mass per sec} \times [V_{1x} - V_{2x}]$$

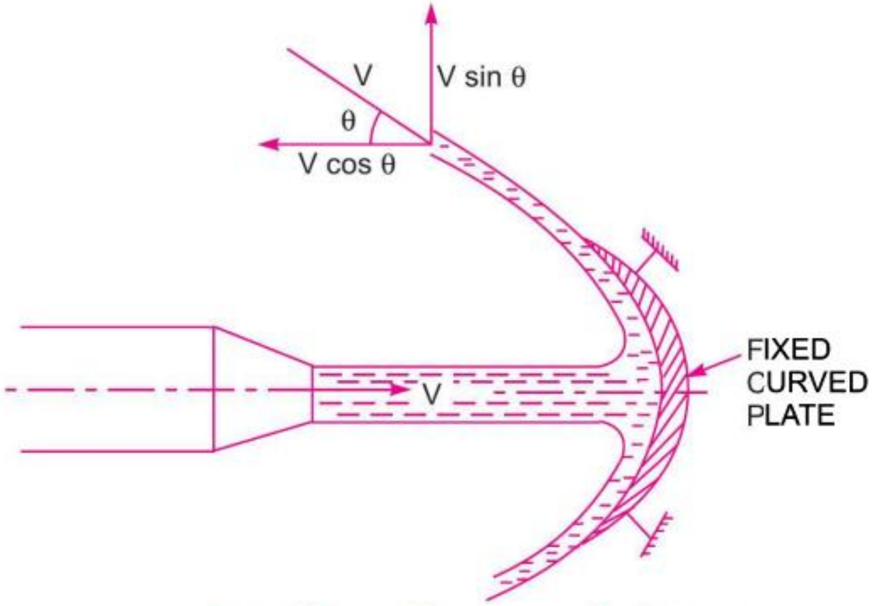
V_{1x} = Initial velocity in the direction of jet = V

V_{2x} = Final velocity in the direction of jet = $-V\cos\theta$

$$F_x = \rho a V[V - (-V\cos\theta)]$$

$$= \rho a V[V + V\cos\theta]$$

$$= \rho a V^2[1 + \cos\theta]$$



Jet striking a fixed curved plate at centre.

$$F_y = \text{Mass per sec} \times [V_{1y} - V_{2y}]$$

V_{1y} = Initial velocity in the direction of $y = 0$

V_{2y} = Final velocity in the direction of $y = V \sin \theta$

$$F_y = \rho a V[0 - V \sin \theta]$$

$$= -\rho a V^2 \sin \theta$$

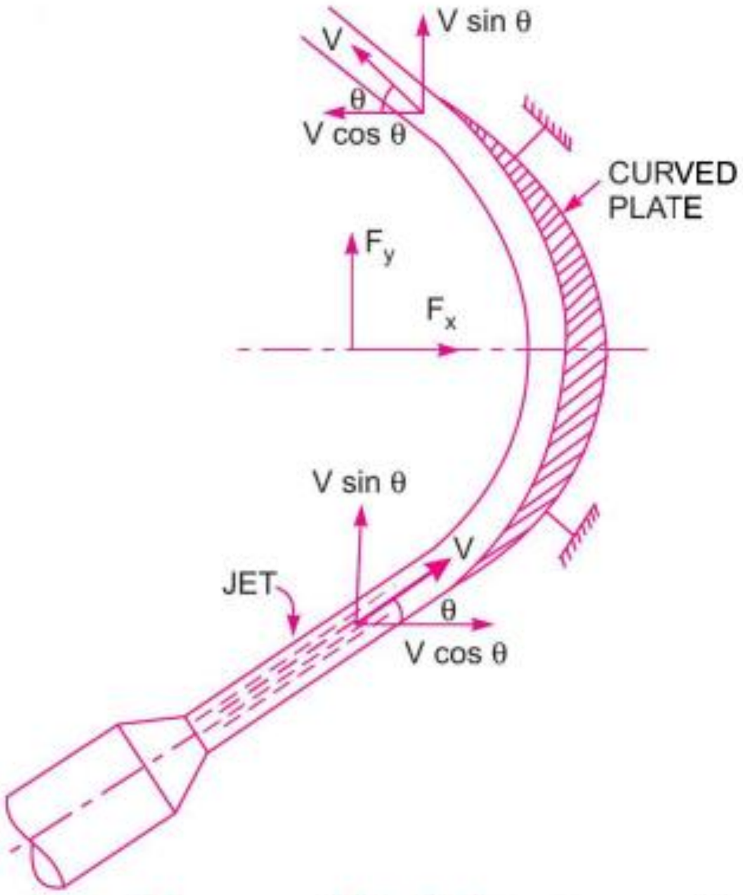
-ve sign means that force is acting in the downward direction. In this case the angle of deflection of the jet = (180° - θ)

(B) Jet strikes the curved plate at one end tangentially when the plate is symmetrical.

Let the jet strikes the curved fixed plate at one end tangentially as shown in Figure.

Let the curved plate is symmetrical about x-axis.

Then the angle made by the tangents at the two ends of the plate will be same.



Jet striking curved fixed plate at one end.

Let V = Velocity of jet of water,

θ = Angle made by jet with x-axis at inlet tip of the curved plate.

If the plate is smooth and loss of energy due to impact is zero, then the velocity of water at the outlet tip of the curved plate will be equal to **V.**

The forces exerted by the jet of water in the directions of x and y are

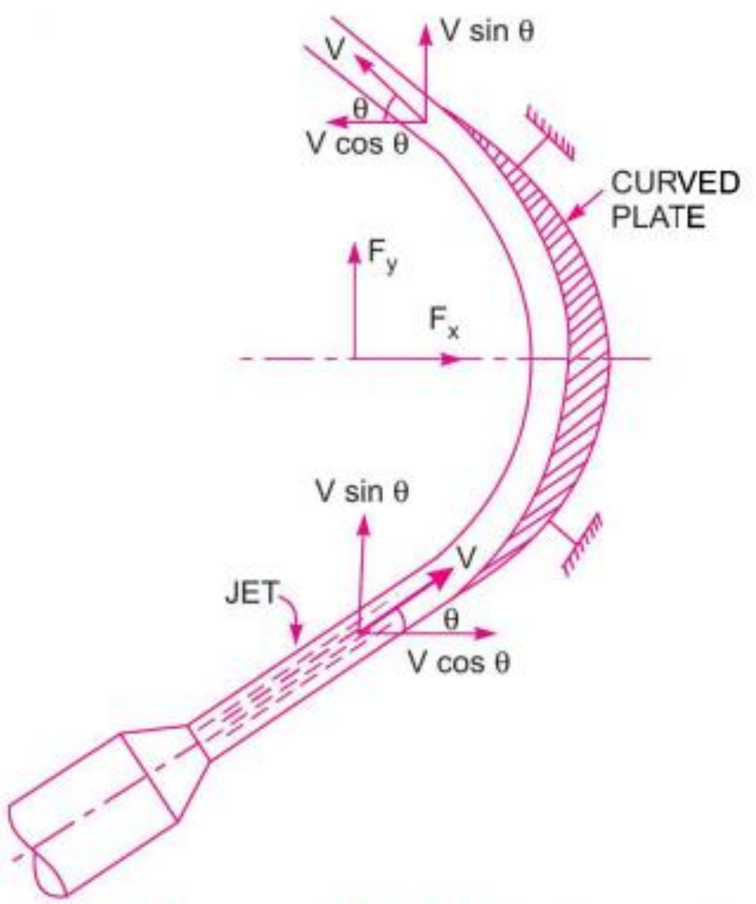
$$F_x = (\text{mass/sec}) \times [V_{1x} - V_{2x}]$$

$$= \rho a V[V \cos \theta - (-V \cos \theta)]$$

$$= \rho a V[V \cos \theta + V \cos \theta]$$

$$= 2\rho a V^2 \cos \theta$$

$$F_y = \rho a V[V_{1y} - V_{2y}] \quad = \rho a V[\, V \sin \theta - V \sin \theta \,] = 0$$



Jet striking curved fixed plate at one end.

(C) Jet strikes the curved plate at one end tangentially when the plate is unsymmetrical.

When the curved plate is unsymmetrical about x-axis, then angle made by the tangents drawn at the inlet and outlet tips of the plate with x-axis will be different.

θ = angle made by tangent at inlet tip with x-axis,

ϕ = angle made by tangent at outlet tip with x-axis.

The two components of the velocity at inlet are

$$V_{1x} = V \cos \theta \text{ and } V_{1y} = V \sin \theta$$

The two components of the velocity at outlet are

$$V_{2x} = -V \cos \phi \text{ and } V_{2y} = V \sin \phi$$

The forces exerted by the jet of water in the directions of x and y are

$$F_x = \rho a V[V_{1x} - V_{2x}]$$

$$= \rho a V[V \cos \theta - (-V \cos \phi)]$$

$$= \rho a V[V \cos \theta + V \cos \phi]$$

$$= \rho a V^2 [\cos \theta + \cos \phi]$$

$$F_y = \rho a V[V_{1y} - V_{2y}]$$

$$= \rho a V[V \sin \theta - V \sin \phi]$$

$$= \rho a V^2 [\sin \theta - \sin \phi].$$

Force on the Inclined Plate Moving in the Direction of the Jet.

Let a jet of water strikes an inclined plate, which is moving with a uniform velocity in the direction of the jet as shown in Figure

Let

V = Absolute velocity of jet of water,

u = Velocity of the plate in the direction of jet,

a = Cross-sectional area of jet, and

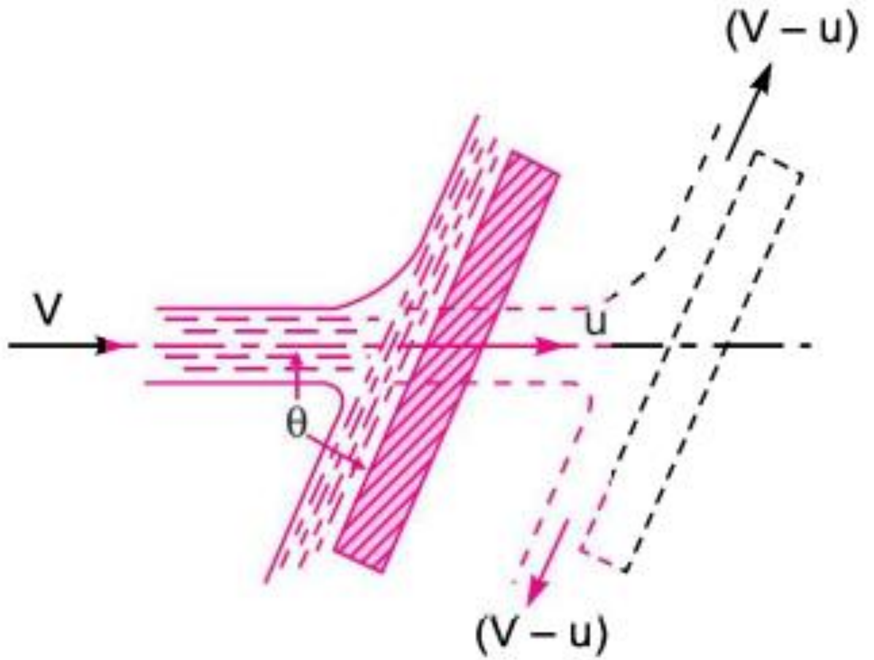
Θ = Angle between jet and plate.

Relative velocity of jet of water = (V - u)

There fore The velocity with which jet strikes = (V - u)

Mass of water striking per second

=p x a x (V - u)



Jet striking an inclined moving plate.

If the plate is smooth and loss of energy due to impact of the jet is assumed zero, the jet of water will leave the inclined plate with a velocity equal to (V– u).

The force exerted by the jet of water on the plate in the direction normal to the plate is given as

F_n = Mass striking per second × [Initial velocity in the normal direction with which jet strikes - Final velocity]

$= \rho a (V - u) [(V - u) \sin\theta - 0] =$

$= \rho a (V - u)^2 \sin\theta$

This normal force F, is resolved into two components namely F_x and F_Y in the direction of the jet and perpendicular to the direction of the jet respectively.

$$F_x = F_n \sin\theta = \rho a (V - u)^2 \sin^2\theta$$

$$F_y = F_n \cos\theta = \rho a (V - u)^2 \sin\theta \cos\theta$$

$\therefore$ Work done per second by the jet on the plate

$= F_x \times$ Distance per second in the direction of x

$= F_x \times u = \rho a(V - u)^2 \sin^2\theta \times u = \rho a (V - u)^2 u \sin^2\theta$ N m/s.

Problem 17.11 *A jet of water of diameter 10 cm strikes a flat plate normally with a velocity of 15 m/s. The plate is moving with a velocity of 6 m/s in the direction of the jet and away from the jet. Find:*

(i) the force exerted by the jet on the plate

(ii) work done by the jet on the plate per second.

Diameter of the jet, $d = 10 \text{ cm} = 0.1 \text{ m}$

$\therefore$ Area, $a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (.1)^2 = .007854 \text{ m}^2$

Velocity of jet, $V = 15 \text{ m/s}$

Velocity of the plate, $u = 6 \text{ m/s}$.

(*i*) The force exerted by the jet on a moving flat vertical plate is given by equation

$$F_x = \rho a (V - u)^2 = 1000 \times .007854 \times (15 - 6)^2 \text{ N} = \mathbf{636.17 \text{ N. Ans.}}$$

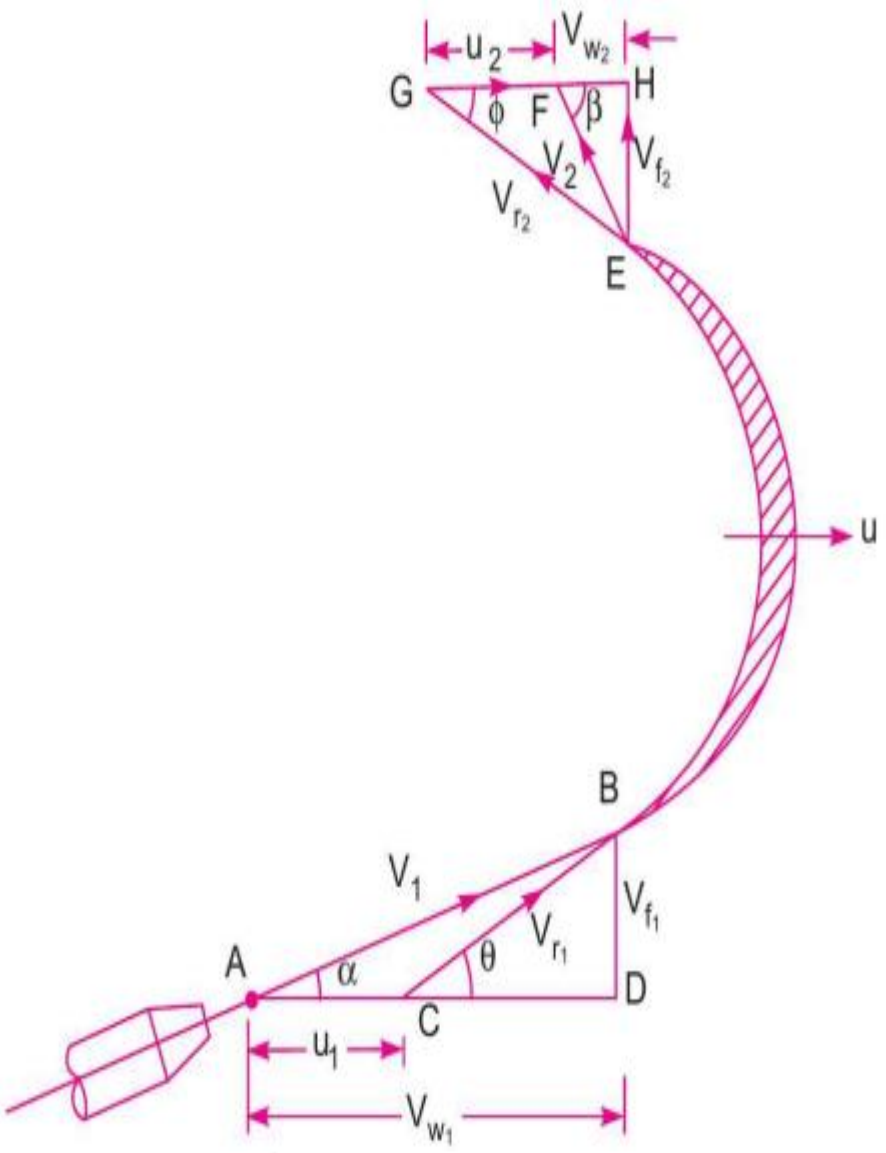
(*ii*) Work done per second by the jet $= F_x \times u = 636.17 \times 6 =$ **3817.02 Nm/s. Ans.**

Force Exerted by a Jet of Water on an Unsymmetrical Moving Curved Plate when Jet Strikes Tangentially at one of the Tips.

Figure shows a jet of water striking a moving curved plate (also called vane) tangentially, at one of its tips.

As the jet strikes tangentially, the loss of energy due to impact of the jet will be zero.

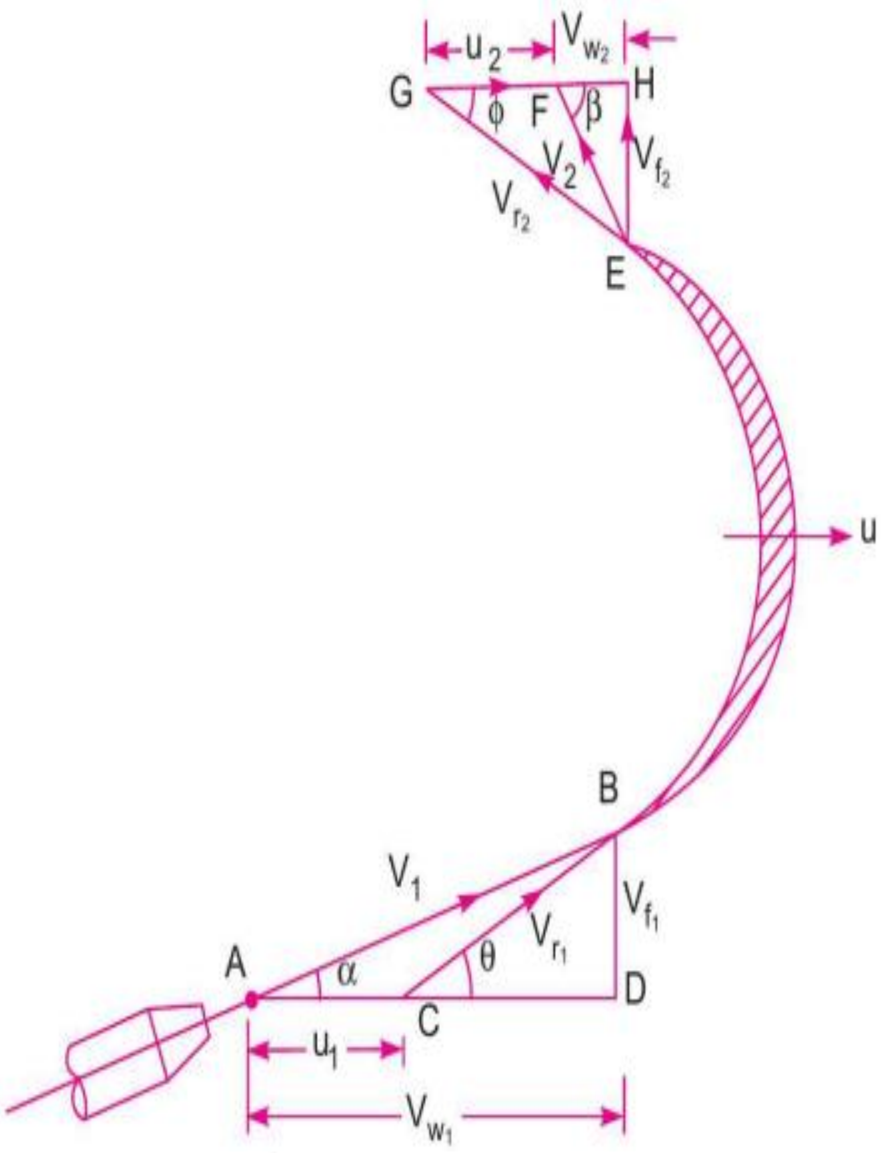
In this case as plate is moving, the velocity with which jet of water strikes is equal to the relative velocity of the jet with respect to the plate



Jet striking a moving curved vane at one of the tips.

Force Exerted by a Jet of Water on an Unsymmetrical Moving Curved Plate when Jet Strikes Tangentially at one of the Tips.

Also as the plate is moving in different direction of the jet, the relative velocity at inlet will be equal to the vector difference of the **velocity of jet** and **velocity of the plate at inlet.**



Jet striking a moving curved vane at one of the tips.

V_1 = Velocity of the jet at inlet.

u_1 = Velocity of the plate (vane) at inlet.

V_{r_1} = Relative velocity of jet and plate at inlet.

α = Angle between the direction of the jet and direction of motion of the plate, also called guide blade angle.

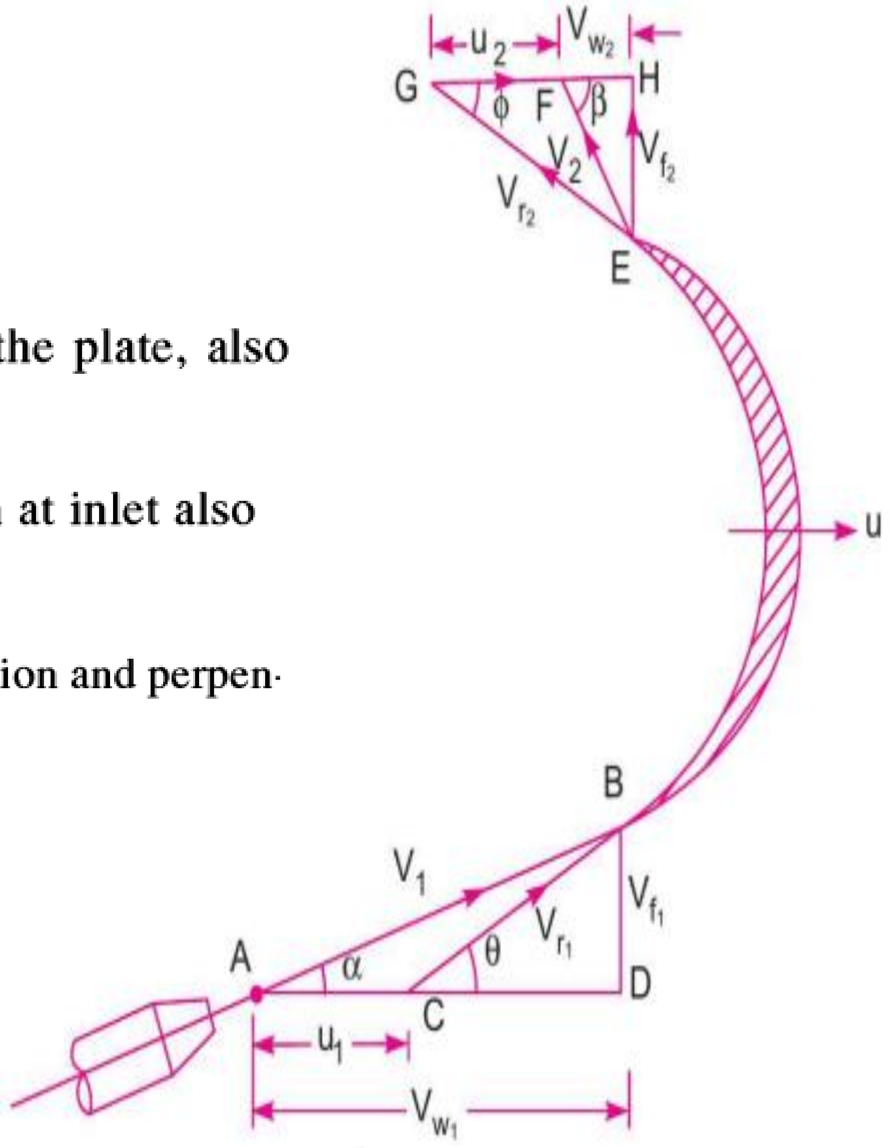
θ = Angle made by the relative velocity (V_{r_2}) with the direction of motion at inlet also called vane angle at inlet.

V_{w_1} and V_{f_1} = The components of the velocity of the jet V_1, in the direction of motion and perpendicular to the direction of motion of the vane respectively.

V_{w_1} = It is also known as velocity of whirl at inlet.

V_{f_1} = It is also known as velocity of flow at inlet.

V_2 = Velocity of the jet, leaving the vane or velocity of jet at outlet of the vane.



Jet striking a moving curved vane at one of the tips.

u_2 = Velocity of the vane at outlet.

V_{r_2} = Relative velocity of the jet with respect to the vane at outlet.

β = Angle made by the velocity V_2 with the direction of motion of the vane at outlet.

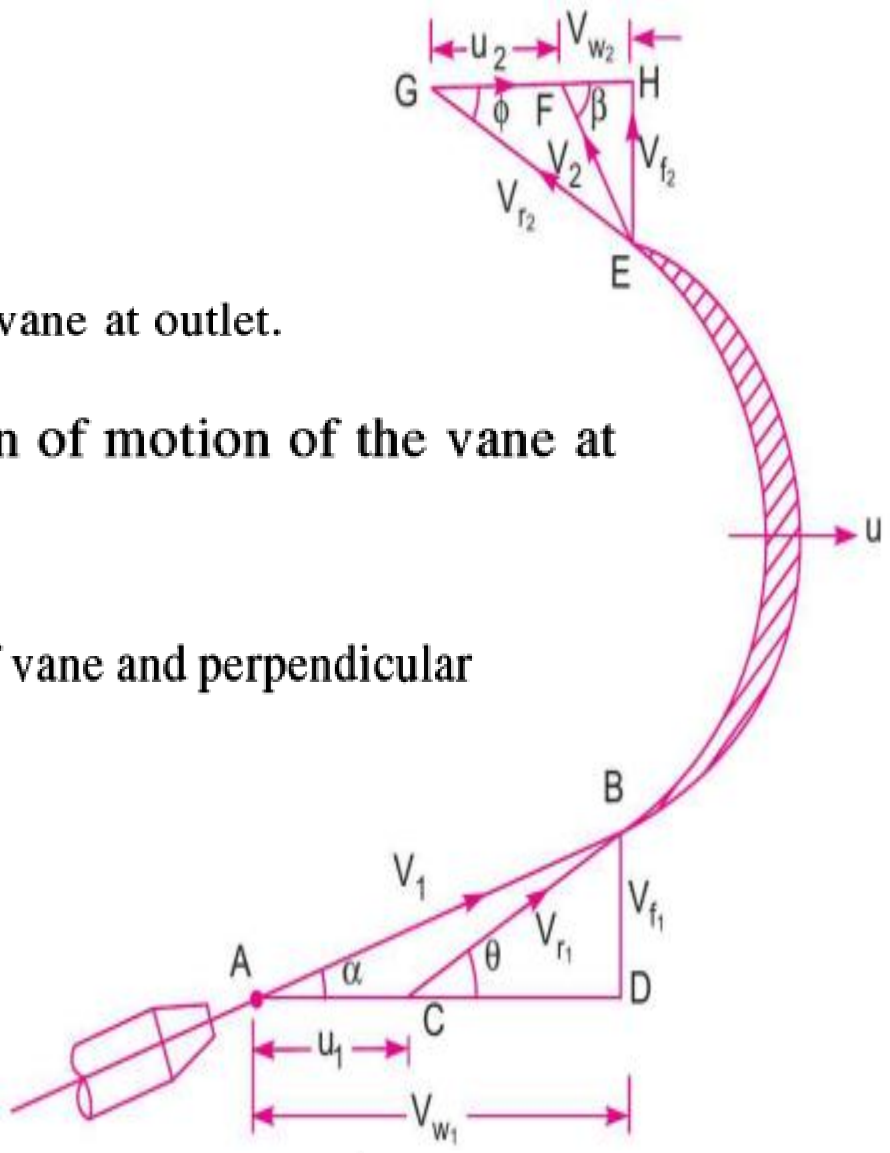
ϕ = Angle made by the relative velocity V_{r_2} with the direction of motion of the vane at outlet and also called vane angle at outlet.

V_{w_1} and V_{f_1} = Components of the velocity V_2, in the direction of motion of vane and perpendicular to the direction of motion of vane at outlet.

V_{w_2} = It is also called the velocity of whirl at outlet.

V_{f_2} = Velocity of flow at outlet.

The triangles *ABD* and *EGH* are called the velocity triangles at inlet and outlet. These velocity triangles are drawn as given below :



Jet striking a moving curved vane at one of the tips.

F_x is written as $F_x = \rho a V_{r_1} \; [V_{w_1} \pm V_{w_2}]$

Work done per second on the vane by the jet

$$= \text{Force} \times \text{Distance per second in the direction of force}$$

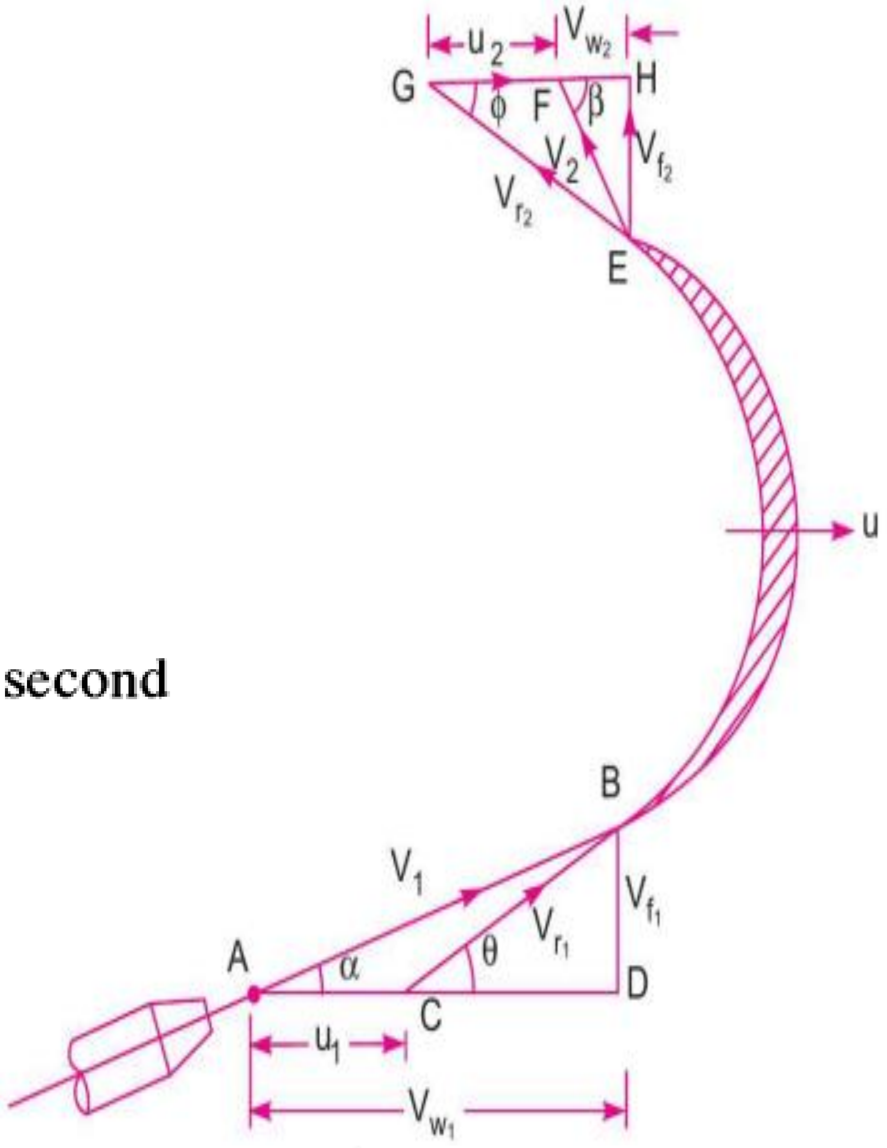
$$= F_x \times u = \rho a V_{r_1} \; [V_{w_1} \pm V_{w_2}] \times u$$

Work done per second per unit weigl $= \dfrac{1}{g} \, [V_{w_1} \pm V_{w_2}] \times u$ Nm/N second

$$= \frac{\rho a V_{r_1} \left[V_{w_1} \pm V_{w_2}\right] \times u}{\text{Weight of fluid striking/s}} \; \frac{\text{Nm/s}}{\text{N/s}}$$

$$= \frac{\rho a V_{r_1} \left[V_{w_1} \pm V_{w_2}\right] \times u}{g \times \rho a V_{r_1}}$$

$$= \frac{1}{g} \, [V_{w_1} \pm V_{w_2}] \times u \text{ Nm/N}$$



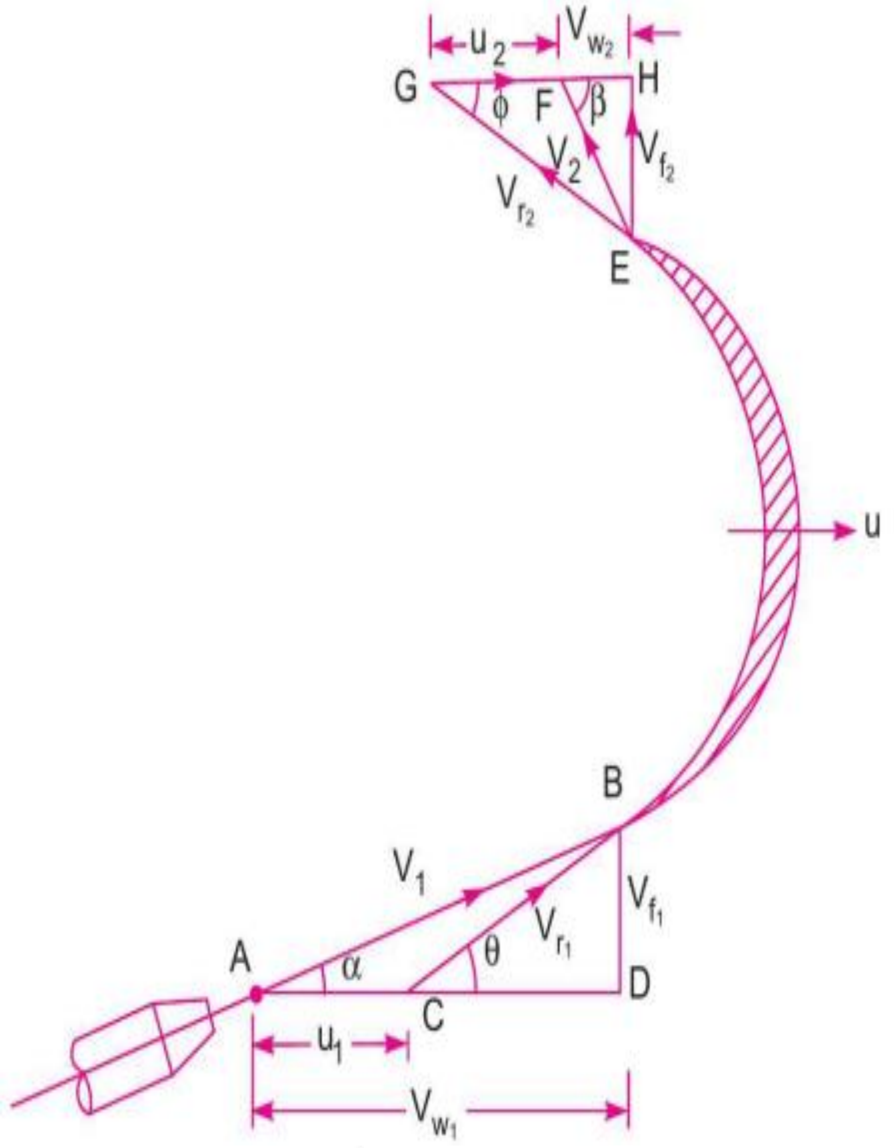
Jet striking a moving curved vane at one of the tips.

Work done/sec per unit mass of fluid striking per second

$$= \frac{\rho a V_{r_1}\left[V_{w_1} \pm V_{w_2}\right] \times u}{\text{Mass of fluid striking / s}} \frac{\text{Nm / s}}{\text{kg / s}}$$

$$= \frac{\rho a V_{r_1}\left[V_{w_1} \pm V_{w_2}\right] \times u}{\rho a V_{r_1}} \text{ Nm/kg}$$

$$= (V_{w_1} \pm V_{w_2}) \times u \text{ Nm/kg}$$



Jet striking a moving curved vane at one of the tips.

3. Efficiency of Jet. The work done by the jet on the vane given by equation (17.20), is the output of the jet whereas the initial kinetic energy of the jet is the input. Hence, the efficiency (η) of the jet is expressed as

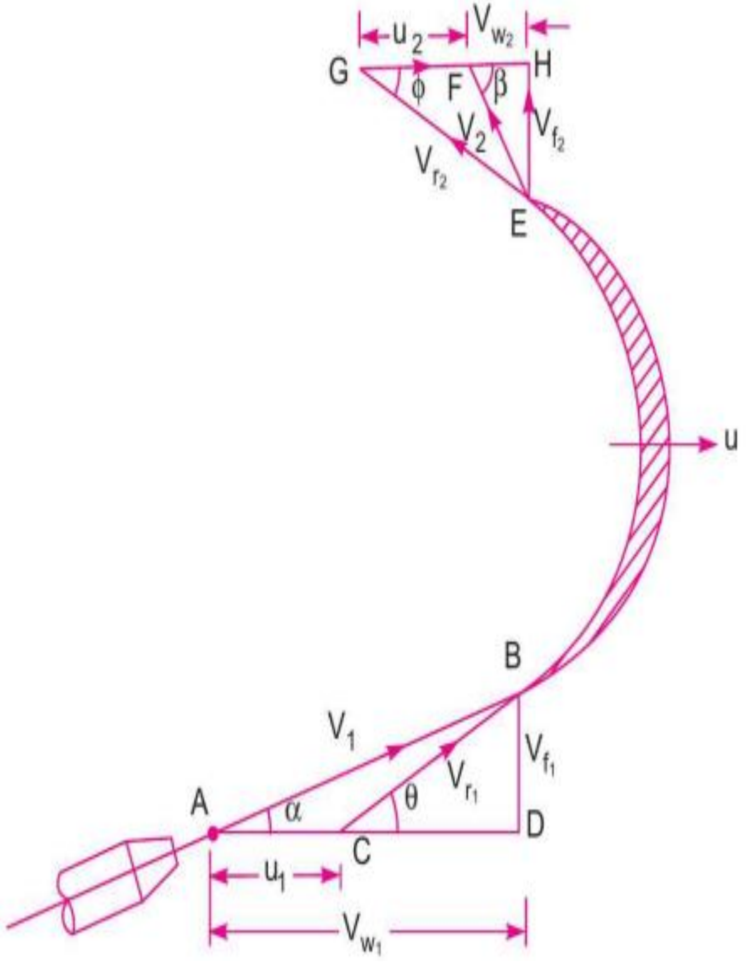
$$\eta = \frac{\text{Output}}{\text{Input}}$$

$$= \frac{\text{Work done per second on the vane}}{\text{Initial K. E. per second of the jet}} \qquad \frac{\rho a V_{r_1} (V_{w_1} \pm V_{w_2}) \times u}{\frac{1}{2} m V_1^2}$$

where m = mass of the fluid per second in the jet = $\rho a V_1$

V_1 = initial velocity of jet

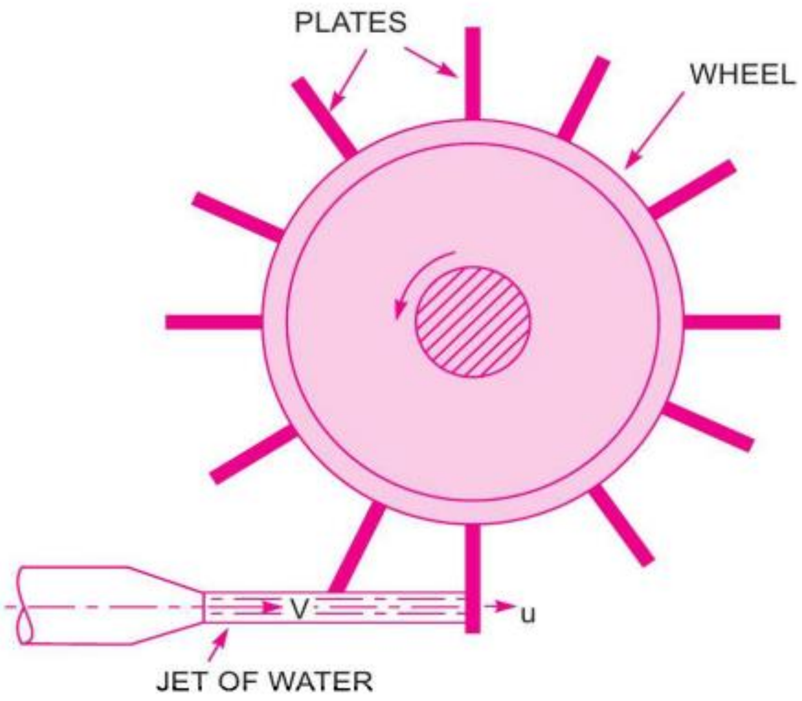
$$\therefore \qquad \eta = \frac{\rho a V_{r_1} \left[V_{w_1} \pm V_{w_2} \right] \times u}{\frac{1}{2} (\rho a V_1) \times V_1^2}.$$



Jet striking a moving curved vane at one of the tips.

Force Exerted by a Jet of Water on a Series of Vanes.

The force exerted by a jet of water on a single moving plate (which may be flat or curved) is not practically feasible. This case is only a theoretical one. In actual practice, **a large number** of plates are mounted on the circumference of a wheel at a fixed distance apart as shown in Figure. The jet strikes a plate and due to the force exerted by the jet on the plate, the wheel starts moving and the 2nd plate mounted on the wheel appears before the jet, which again exerts the force on the 2nd plate. Thus each plate appears successively before the jet and the jet exerts force on each plate. The wheel starts moving at a constant speed.



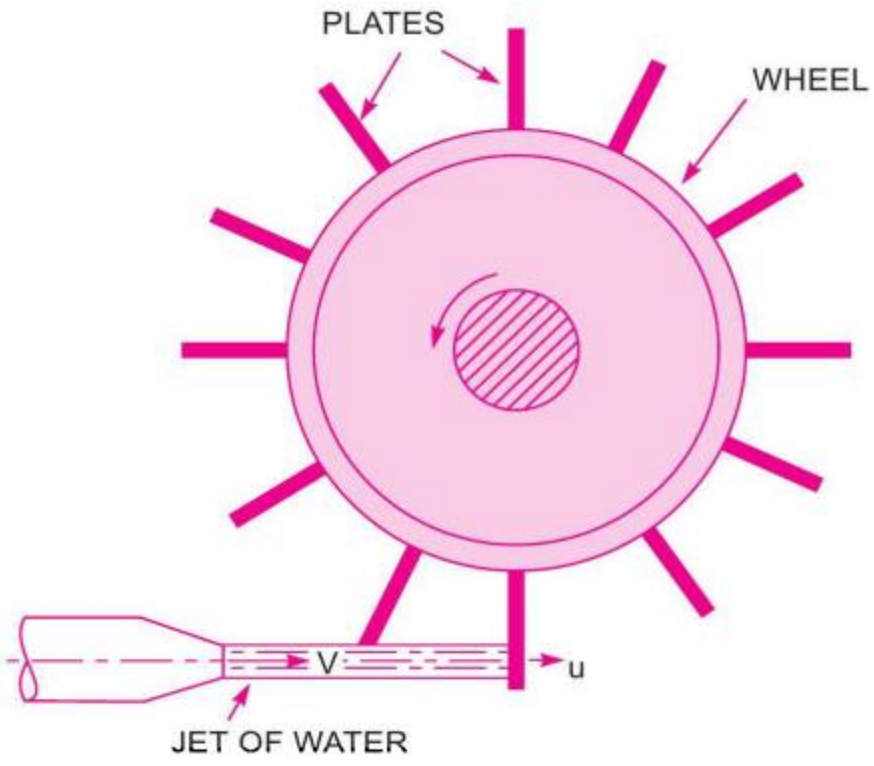
Jet striking a series of vanes.

V = Velocity of jet,

d = Diameter of jet,

a = Cross-sectional area of jet, $= \frac{\pi}{4} d^2$

u = Velocity of vane.



Jet striking a series of vanes.

In this case the mass of water coming out from the nozzle per second is always in contact with the plates, when all the plates are considered. Hence mass of water per second striking the series of plates = $\rho a v$.

Also the jet strikes the plate with a velocity = (V - u).

After striking, the jet moves tangential to the plate and hence the velocity component in the direction of motion of plate is equal to zero.

The force exerted by the jet in the direction of motion of plate,

$$F_x = \text{Mass per second [Initial velocity – Final velocity]}$$

$$= \rho a V[(V - u) - 0] = \rho a V[V - u]$$

Work done by the jet on the series of plates per second

$$= \text{Force} \times \text{Distance per second in the direction of force}$$

$$= F_x \times u \quad = \rho a V[V - u] \times u$$

Kinetic energy of the jet per second

$$= \frac{1}{2} mV^2 = \frac{1}{2} (\rho a V) \times V^2 = \frac{1}{2} \rho a V^3$$

Efficiency, $$\eta = \frac{\text{Work done per second}}{\text{Kinetic energy per second}} = \frac{\rho a V[V - u] \times u}{\frac{1}{2}\rho a V^3} = \frac{2u[V - u]}{V^2}$$

Condition for Maximum Efficiency. For a given jet velocity V, the efficiency will be maximum when

$$\eta = \frac{\text{Work done per second}}{\text{Kinetic energy per second}} = \frac{\rho a V[V-u]\times u}{\frac{1}{2}\rho a V^3} = \frac{2u[V-u]}{V^2}$$

$$\frac{d\eta}{du} = 0$$ Or $$\frac{d}{du}\left[\frac{2u(V-u)}{V^2}\right]$$ or $$\frac{d}{du}\left[\frac{2uV-2u^2}{V^2}\right] = 0$$

$$\frac{2V-2\times 2u}{V^2} = 0 \qquad 2V-4u = 0$$

$$u = \frac{V}{2}.$$

Maximum Efficiency.

Substituting the value of $V = 2u$

$$\eta_{max} = \frac{2u\,[2u - u]}{(2u)^2}$$

$$= \frac{2u \times u}{2u \times 2u} = \frac{1}{2} = 0.5 \text{ or } 50\%.$$